Time-to-Build Echoes
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Abstract

In an optimal growth model with a time–to–build delay, the feasibility condition is a delayed-differential equation, and the Euler-type condition is an advanced-differential equation. As in Kalecki's theory of business cycles, the delayed nature of the feasibility constraint naturally induces cycles. However, the advanced nature of the Euler-type equation dampens fluctuations through smoothing, making the economy converge by oscillations. We refer to this result as time–to–build echoes.

JEL codes: O40, E32, C63

Key words: Time–to–build, Delay, Shooting method, DDEs

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1 Introduction

The claim that delays can induce cycles was emphasized by Kalecki (1935), who models them in continuous time by mean of delayed-differential equations (DDEs).[1] In this note, an optimal growth model with a time–to–build delay is solved by direct application of optimal control theory (see Kolmanovskii and Myshkis (1998)).

When the investment process involves a time–to–build delay, say d, present production depends on past capital (the one produced before $t-d$), making the feasibility condition take the form of a DDE, as in Asea and Zak (1999). Contrary to these authors, we show that the optimal allocation verifies an Euler-type differential equation with advanced terms. The trade-off is then between consumption at time t and consumption at time $t+d$, since resources allocated to investment today will be available for production after the delay has elapsed. This is not a minor point, and makes the resolution of this problem an important, open issue.[2]

The delayed nature of the feasibility constraint naturally induces cycles. This is the fundamental mechanism in Kalecki's theory of business cycles. However, the advanced nature of the Euler-type equation dampens fluctuations: with the aim of smoothing consumption, optimal saving fluctuates to compensate cycles induced by the time–to–build delay. As a result, the economy converges by oscillations. We refer to this result as time–to–build echoes. To illustrate the importance of this mechanism, we propose a numerical algorithm to overcome the simultaneous occurrence of advanced and delayed time arguments in the optimality conditions of growth models with delays. We find that increasing the size of the delay considerably alters the internal dynamics of the model: high values of the delay display persistent oscillations, whereas low values exhibit as smooth dynamics as in the model without time–to–build.

Section 2 presents the model with time–to–build and characterizes optimal solutions. Section 3 implements the algorithm and discusses the properties of the model. A last section concludes.

[1] An ordinary differential equation is an equation connecting the values of an unknown function and some of its derivatives for one and the same time argument, e.g. $F(t, x(t), \dot{x}(t)) = 0$. A DDE involves also the unknown function and some of its derivatives for some past argument, e.g. $F(t, x(t), x(t-d), \dot{x}(t)) = 0$, where the delay $d > 0$ is a constant. The previous equation is said to be an advanced-differential equation if $d < 0$.

[2] The study of optimal growth models with delays largely remains unexplored as the resulting differential equation system involves both DDEs and ADEs, and most of the available methods restrict to DDEs alone. See Boucekkine at al. (2005)

2 The optimal growth model with time–to–build

Let us consider an economy populated by infinitely–lived households with unit aggregate measure. A social planner chooses the path of both consumption and investment so as to maximize the infinite stream of discounted instantaneous utilities subject to the resource constraint, that is

$$\max \int_0^\infty \frac{c(t)^{1-\sigma}-1}{1-\sigma}\ \mathrm{e}^{-\rho t}\ \mathrm{d}t \text{ with } \rho,\sigma > 0,$$

subject to

$$\dot{k}(t) = Ak(t-d)^\alpha - \delta k(t-d) - c(t) \tag{1}$$

with initial conditions $k(t) = k_0(t)$ for all $t \in [-d, 0]$. $k_0(t)$ is the initial capital function, which is taken as given by the social planner. $d > 0$ is the time–to–build lag — *i.e.* machines produced at time t are available for production at time $t + d$, such that the production function is given by $Ak(t-d)^\alpha$, $A > 0$ and $\alpha \in]0, 1[$. Machines depreciate at the exponential rate $\delta > 0$.

The necessary conditions associated to this problem are[3]

$$c(t)^{-\sigma}\ \mathrm{e}^{-\rho t} = \phi(t)$$

$$-\phi(t+d)\left(\alpha\ Ak(t)^{\alpha-1} - \delta\right) = \dot{\phi}(t),$$

and the transversality conditions,

$$\lim_{t\to\infty} \phi(t) \geq 0 \qquad \text{and} \qquad \lim_{t\to\infty} \phi(t)k(t) = 0$$

where $\phi(t)$ is the co-state representing the marginal value of capital produced at time t. Contrary to Asea and Zak (1999), the optimal allocation verifies the following Euler-type differential equation with an advanced term

$$\frac{\dot{c}(t)}{c(t)} = \frac{1}{\sigma}\left(\left(\alpha\ Ak(t)^{\alpha-1} - \delta\right)\left(\frac{c(t)}{c(t+d)}\right)^\sigma \mathrm{e}^{-\rho d} - \rho\right), \tag{2}$$

where the marginal productivity of capital available at time $t + d$ is weighted by the ratio of the marginal utility of consumption at $t + d$ to the marginal utility of consumption at t. Through investment, the social planner substitutes consumption at t for consumption at $t + d$. Notice that the optimal

[3] See Kolmanovskii and Myshkis (1998).

conditions converge to the solution of the standard neoclassical growth model when $d \to 0$.

Let us define a steady state as an optimal allocation for which $\dot{c}(t) = \dot{k}(t) = 0$. In this case, the capital stock k_s and consumption c_s are given by

$$\begin{aligned} k_s &= \left(\frac{\alpha A}{\rho\ \mathrm{e}^{\rho d} + \delta}\right)^{\frac{1}{1-\alpha}} \\ c_s &= k_s^{\alpha} - \delta k_s. \end{aligned}$$

where the only difference with respect to the standard neoclassical growth model relies on the ratio of marginal utilities, which is represented by the term $\mathrm{e}^{\rho d}$. Consequently, the steady state is unique. Next, we discuss our numerical results and the dynamic properties of the time–to–build model.

3 Solving for the short-run dynamics

System (1)–(2) is a mixed delayed differential equation (MDDE) system, with initial condition $k(t) = k_0(t)$ for all $t \in [-d, 0]$. To illustrate the dynamic properties of this system, we implement an algorithm that combines *(i)* a standard method of steps [cf. Paul (1997)] and *(ii)* a shooting method [cf. Judd (1998)]. An sketch of the algorithm is in the Appendix.

Figure 1 reports the dynamics of the economy for different values of the delay, d, when the economy experiments an unexpected 5% negative shock on its steady state capital level. The choice of parameter values is given in Table 1. These parameter values are standard in the growth literature and fully determine the steady state of the model, in Table 2. All reported dynamics are expressed in percentage deviation from the steady state and are obtained using a polynomial approximation of the consumption expectation function.[4] The grey line corresponds to the standard optimal growth model ($d = 0$) which will be used as a benchmark, the dark dashed line corresponds to a short time–to–build model ($d = 2$) and finally the dark plain line refers to a long time–to–build situation ($d = 20$).

As is well–known, the standard optimal growth model displays monotonic convergence to the steady state. Since the initial capital stock is lower, output drops while the real interest rate rises. This triggers an instantaneous decrease in consumption both by wealth and intertemporal substitution motives. Conversely, the increase in the real interest rate creates an incentive

[4] We chose a basis of Chebychev polynomials. The order of approximation is set to $n = 20$. Results do not significantly differ from those obtained using a pointwise approximation.

to accumulate and investment rises, therefore increasing the pace of capital accumulation. Increase in the capital stock puts downward pressure on the interest rate and enables an increase in output. Both effects make the household consume more while the decrease in the marginal efficiency of capital triggers a slowdown in investment. This makes the economy to converge back to the steady state monotonically.

A long delay considerably alters the internal dynamics of the model. With a time–to–build lag, the economy is stuck with an output level of k_0 for the time span $(0, d]$, and any extra investment will only become productive in period d. Setting $d = 20$, the convergence path is not monotonic anymore but rather displays dampening oscillations due to echo effects. Since capital is lower than its steady state value, it is optimal for the household to increase its investment effort so as to fasten the pace of accumulation. But, because of time–to–build, output is stuck at a low level for a long period of time. This makes the household to spread her investment effort over a large period of time to smooth consumption. As capital accumulation takes place, household expectations concerning future interest rate are downward sloping, such that as time approaches period d, investment becomes less and less attractive — even becoming lower than its steady state value. Consequently, the household can consume more, which brings back consumption closer to its steady state level. When accumulated capital in period 0 becomes operative (period d), investment is at its lower level — 6% below its steady state. This slowdown in the pace of accumulation makes it beneficial for the household to raise her investment effort. Investment then starts increasing again, triggering a slowdown in increase in consumption. But once again, the faster pace of accumulation exerts negative pressures on the interest rate that weaken household's desire to invest. Investment then starts declining until period $2d$ (period 40) allowing for greater increase in consumption. This oscillating process takes place until convergence of the economy to its steady state value.

The model with a short–horizon time–to–build looks almost identical to the standard optimal growth model, in that it also converges monotonically to the steady state. With a time–to–build of length 2, the economy is stuck for only 2 periods and investment responds more (2% for $d = 2$ to be compared to the earlier 1.58% in the case $d = 20$, see Table 3). Output being given, consumption drops by a higher amount (nearly 2.4% to be compared to 2% when $d = 20$, see again Table 3). As soon as time reaches period d, new capital becomes productive, and output suddenly rises while decreasing returns bring the real interest rate down. As can be seen from the Euler equation defining the household's consumption/saving behavior, this makes it possible to smooth consumption. From period d on, the dynamics of the economy is close to that of the optimal growth model albeit smoother because of the time–to–build hypothesis.

4 Concluding remarks

This note shows that in the case of a time–to–build lag, as well as in other growth models with delays, the forward–looking behavior of agents yields advanced terms which tend to mitigate the cycles found by Kalecki and others. To overcome the simultaneous occurrence of advanced and delayed time arguments in the optimality conditions of growth models with delays we further propose a simple shooting algorithm. This serves to give new insight on the role of advanced time arguments to mitigate the cycles induced by lag structures.

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Table 1: Parameters

Discount rate	ρ	0.05
Intertemporal substitution	$1/\sigma$	2/3
State of technology	A	1
Capital elasticity	α	0.3
Depreciation rate	δ	0.1

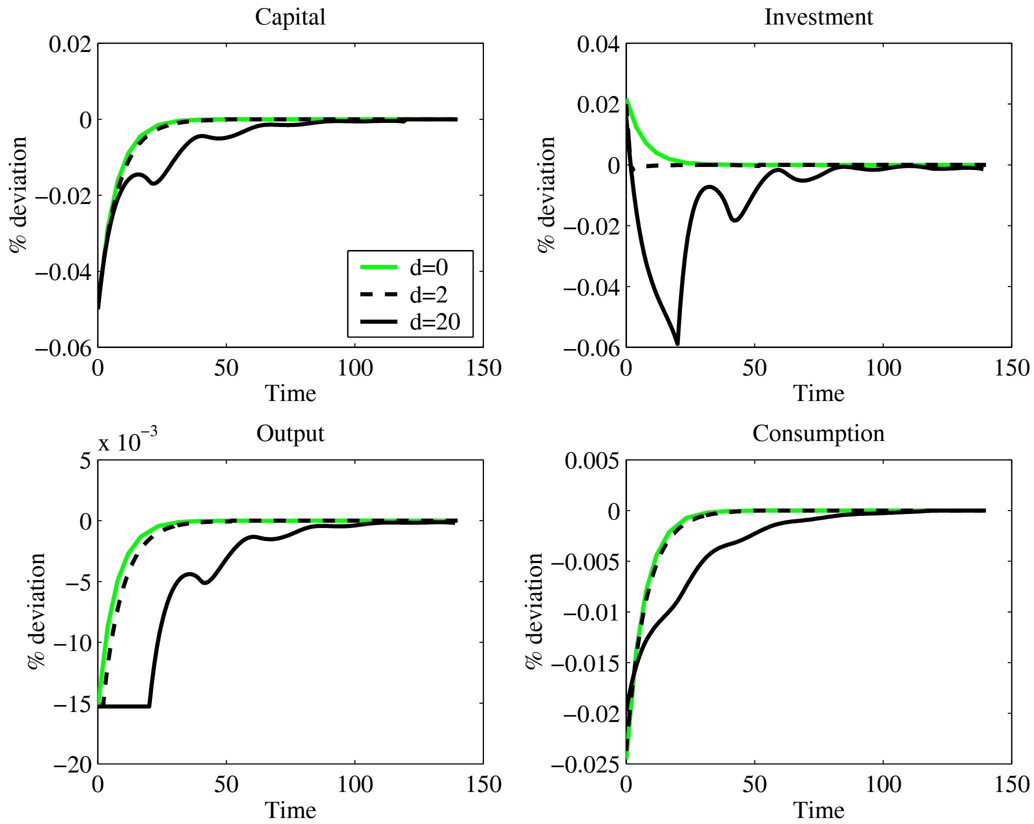
Table 2: Steady state

	Ramsey (d=0)	T–to–B (d=2)	T–to–B (d=20)
k_s	2.6918	2.5625	1.4096
c_s	1.0767	1.0699	0.9675

Table 3: Impact effect (% deviation from steady state)

	Ramsey (d=0)	T–to–B (d=2)	T–to–B (d=20)
k	-0.0500	-0.0500	-0.0500
c	-0.0245	-0.0237	-0.0198
i	0.0218	0.0200	0.0158
y	-0.0153	-0.0153	-0.0153

Figure 1: Dynamics



Appendix: An Algorithm for MDDE systems

An outline of the algorithm used to compute the dynamics is the following:

Set an initial guess for the consumption expectation $\widetilde{c}_0(t+d)$. Set a time span $[0;T]$, $T = S \times d$, where S is the number of steps and d is the delay. Set an initial condition for the consumption path: bracketing over an upper bound $c_H > 0$ (the solution diverges from below) and a lower bound $c_L \geqslant 0$ (or above). Set the iteration counter ℓ to 0, and stopping criteria $\varepsilon^e > 0$ (for the expectation guess) and $\varepsilon^s > 0$ (for the shooting part).

Step 1 (Solving the system conditional on an expectation function): Given an expectation guess, $\widetilde{c}_\ell(t)$, perform a shooting algorithm.

1. Set $c_0 = (c_H + c_L)/2$
2. Given c_0 and $\widetilde{c}_\ell(t)$, solve the dynamic system by the method of steps:
   (a) Set $k(t) = k_0(t)$ for $t \in (-d, 0]$. Set $i = 0$
   (b) Given $k_i(t)$, $t \in ((i-1) \times d, i \times d]$, $\widetilde{c}_\ell(t)$, and initial c_0 solve
   $$\begin{aligned} \dot{k}(t) &= Ak_i(t-d)^\alpha - c(t) - \delta k_i(t-d) \\ \dot{c}(t) &= \frac{c(t)}{\sigma}\left(\mathrm{e}^{-\rho d}\left(\alpha A k(t)^{\alpha-1} - \delta\right)\left(\frac{c(t)}{\widetilde{c}_\ell(t+d)}\right)^\sigma - \rho\right) \end{aligned}$$
   and set $k_i(t) = k(t)$ for $t \in (i \times d, (i+1) \times d]$.
   (c) Set $i = i+1$ and go back to (b) until $i = S$.
3. If $|c(t) - c_s| < \varepsilon^s$ for $t \in ((S-1) \times d, S \times d)$ stop; else
   - If $c(t) > c_s$ for $t \in ((S-1) \times d, S \times d)$, set $c_H = c_0$ and go to 1.
   - If $c(t) < c_s$ for $t \in ((S-1) \times d, S \times d)$, set $c_L = c_0$ and go to 1.

Step 2 (Revision of expectations): If $\|c(t) - \widetilde{c}_\ell(t)\| < \varepsilon^e$, stop, else revise the expectation function and go back to 1.

The revision of the expectation function depends on the use of the approximation procedure.

Details on the practical implementation of the algorithm are discussed in a technical appendix which is available upon request.