fedea

Fundación de
Estudios de
Economía Aplicada

The Measurement of Consensus: An Axiomatic Analysis*

by

Jorge Alcalde–Unzu **

Marc Vorsatz ***

DOCUMENTO DE TRABAJO 2008-28

July 2008

* We are very grateful to the seminar audiences in Bilbao, Montreal, Soria, and Vigo.

** Universidad del País Vasco.

*** FEDEA.

ISSN:1696-750X

Jorge Juan, 46
28001 Madrid -España
Tel.: +34 914 359 020
Fax: +34 915 779 575
infpub@fedea.es

N.I.F. G-78044393

The Measurement of Consensus: An Axiomatic Analysis*

Jorge Alcalde–Unzu† and Marc Vorsatz‡

July 29, 2008

Abstract

The cohesion of a society depends to large extend on the degree to which its members coincide in their preferences (the *consensus*). This paper proposes axioms a consensus measure should satisfy from a normative point of view and characterizes first a class of linear and additive measures which fulfills an ordinal property similar to the concepts of first order stochastic dominance in the literature on individual decision making under risk and Lorenz curve domination in the literature on income inequality measurement. With the help of some additional properties, it is then possible to isolate from this broad class of measures a subfamily that only depends on a single parameter. Finally, we show that the consensus measures associated with the focal parameters of this subfamily have an intuitive explanation and we characterize them separately.

1 Introduction

Consensus Measures In the classical problem of social choice the objective of a group of individuals is to order a given set of alternatives. Examples include, among others, political elections, the location of public goods, and strategic firm planning. The cohesion in the society and the probability that conflicts between its members arise depends to large extend on the degree to which individuals coincide in their preferences. If their opinions are very similar, the group is rather united, otherwise it tends to be segregated. In order to better understand this source of conflict, we aim at evaluating the similarities of individual preferences (the

*We are very grateful to the seminar audiences in Bilbao, Montreal, Soria, and Vigo.

†Corresponding author. Departamento de Economía Aplicada IV, Universidad del País Vasco, Avenida Lehendakari Agirre 83. 48015 Bilbao, Spain. E-mail: jorge.alcalde@ehu.es. Financial support from the Spanish Ministry of Education and Science, through the Juan de la Cierva program and project SEJ2006-11510, is gratefully acknowledged.

‡Fundación de Estudios de Economía Aplicada (FEDEA), Calle Jorge Juan 46, 28001 Madrid, Spain. Email: mvorsatz@fedea.es. Financial support from the Spanish Ministry of Education and Science, through the Ramón y Cajal program, is gratefully acknowledged.

consensus) in the same way as the alikeness of individuals in terms of other aspects related to group cohesion, such as income, is measured.

The main objective of the paper is precisely this: to study axiomatically the possibility of constructing a numerical function that evaluates the consensus in a group of individuals. To be more exact, we suppose that each individual has a linear ranking that reflects her/his preference on the set of alternatives. A group of individuals is represented by a preference profile, a list of individual preferences. With this primitives at hand, we aim at constructing a measure that assigns to each preference profile of every possible society (the group of individuals is allowed to vary) a value from the unit interval, satisfying the implicit assumption that higher values imply more consensus.

Contribution Following a normative approach, we propose a set of properties that a consensus measure should meet. The first property, *Anonymity*, is a classical property stating that the preferences of all individuals are equally important. The second property, *Independence of Irrelevant Pairs*, is the parallel of Arrow's [3] Independence of Irrelevant Alternatives adapted to our context. It states that the effect of a change of the opinion over only two alternatives should not depend on how the individuals order other pairs of alternatives. According to the third axiom, *Neutrality*, the consensus measure should not be biased with respect to certain alternatives. Finally, *Monotonicity* says that if an individual only changes her/his opinion on how to order one particular pair of alternatives, the change in the consensus depends on which of these two alternatives wins in a majority voting when all individuals from the group apart from the one who changes preferences participate in the election. If the alternative that is made better off with the change wins (ties) in the pairwise comparison, the consensus should increase (remain constant).

We prove in Theorem 1 that this set of properties characterizes a class of linear and additive measures which satisfies an ordinal dominance criterion similar to the ones of first order stochastic dominance (see, among others, Rothschild and Stiglitz [16]), and Lorenz curve domination (see, Atkinson [2]). This family of consensus measures, called Ω, calculates for every pair of alternatives the absolute difference of how many individuals prefer one alternative and how many prefer the other. The absolute differences are then aggregated by a weighted sum. As a consequence, the highest consensus is obtained if all individuals from a society agree on all pairwise comparisons (they have exactly the same preferences), whereas the lowest possible consensus corresponds to the preference profiles where for all pairs of alternatives, one half of the population prefers one and the other half prefers the other alternative.

Since these four properties only impose a partial order on the set of all preference profiles and the class Ω is rather large, we add three additional properties with the objective of restricting this class of measures further. *Replication* requires that if a group of individuals together with its opinions is cloned a number of times, the consensus in the bigger group containing all the clones should be equal to that of the original group. *Full Range* asks that the minimal and maximal values of the consensus measure (0 and 1) should be attained for some preference profiles of possibly different groups of individuals. Adding these two properties together with *Consistency* –the evaluation of successive changes should be done without discontinuities– to the former ones implies that the vector of weights identified in Theorem 1 depends on a single parameter (Theorem 2). This parameter ($\gamma \in [0.25, 0.75]$) evaluates the effect on the consensus whenever one individual changes her/his opinion on a single pair of alternatives as a function of how many of the remaining individuals in the society prefer one alternative and how many the other. If $\gamma \in (0.50, 0.75]$, more importance is given to changes when the society is divided on that pair of alternatives. If $\gamma \in [0.25, 0.5)$, changes that correspond to situations when the society clearly favors one of the two alternatives receive more weight. Finally, if $\gamma = 0.5$, all changes are evaluated equally.

In the last part of the paper, we analyze the focal cases of the family Γ characterized in Theorem 2. First, we concentrate on the case when $\gamma = 0.25$, a measure that we will call *corrected normalized average tau* ($\hat{\tau}$). Leaving apart some minor adjustments, it calculates for any pair of individuals, the Kemeny distance (in relative terms) between their rankings. Then, the average of all relative distances over all pairs of individuals is determined (see, Hays [11]). Finally, the measure is normalized to the unit interval. From a formal point of view, $\hat{\tau}$ is characterized by strengthening Independence of Irrelevant Pairs and Consistency to *Proportionality.* This condition states that the increase in the consensus changes proportionally with the number of individuals that prefer the alternative that is made better off (Theorem 3). The measure associated with $\gamma = 0.75$ is closely linked to $\hat{\tau}$ because it is obtained when the proportionality condition is inverted.

In order to characterize the measure associated with $\gamma = 0.5$, which we will call *average* *sigma* ($\bar{\sigma}$), Independence of Irrelevant Pairs together with Consistency is strengthened to *Strong Independence of Irrelevant Pairs* (Theorem 4). This property says that the effect on the consensus of a change in the opinion of an individual over two alternatives only depends on which alternative wins according to a majority election on these two alternatives. Intuitively, $\bar{\sigma}$ can be calculated as follows: for any pair of alternatives, take the absolute difference between the number of individuals who prefer one alternative and the number of individuals who prefer the other. Then, calculate the average of all absolute differences over all possible pairs of alternatives and divide it by the number of individuals in the society.

Relation to the Literature The concept of a consensus measure has not been object to a broad study. Among others, Kemeny [13], Cook and Seiford [6], and Klamler [14] have proposed some distance measures between rankings for the specific case of groups of only two individuals. For the general case of an arbitrary group size, Bosch [4] pioneered the concept of a consensus measure. Apart from his work, Meskanen and Nurmi [15] study the relationship between some classical social welfare functions and distance functions between rankings and profiles. They find that for each social welfare function, the social ranking is the nearest to the preference profile with respect to some distance. As a result, they defend that this distance can be interpreted as the consensus of the group. However, this approach is indirect, since it depends on the social welfare function selected in the first phase.

Remainder We proceed as follows: in the next section, we introduce basic notation and definitions. Section 3 presents the first set of axioms and the characterization of Ω. We also include a graphical interpretation of these measures. Section 4 introduces some additional properties and provides the characterization of Γ. Section 5 studies the focal measures of this family. We conclude in Section 6 with some remarks. The independence of the properties is established in the Appendix.

2 Notation and Definitions

Consider a finite set of alternatives X of cardinality $k \geq 2$ and a (countable) infinite set of individuals denoted by the set of integers $\mathbb{N}$. Our objective is to evaluate the consensus for all finite societies $N \subset \mathbb{N}$ of size $n \geq 2$. Elements of X are usually denoted by x, y, and z, generic individuals are indexed by i and j. We will also make frequent use of the letters A, B, and C to denote finite sets of individuals. Let $\mathbb{Q}$ be the set of rational numbers. Given any rational number $r \in \mathbb{Q}$, $[r] \in \mathbb{N}$ refers to the largest integer lower than or equal to r.

Let P_i be the strict preference relation of individual i on X. We assume that P_i is complete, transitive, and antisymmetric. The set of all strict preference relations on X is denoted by $\mathcal{P}$. A *profile* $P = (P_i)_{i\in\mathbb{N}} \in \mathcal{P}^{\mathbb{N}}$ is a list of all individual preference relations. Given a profile $P \in \mathcal{P}^{\mathbb{N}}$ and a society $N \subset \mathbb{N}$, a *preference profile* $P_N = (P_i)_{i\in N} \in \mathcal{P}^N$ is an n–tuple of individual preference relations, one for every individual belonging to N. We also say that two preference profiles $P_A \in \mathcal{P}^A$ and $P'_B \in \mathcal{P}^B$, corresponding to the societies A and B of equal size, are *isomorphic* whenever there exists a one–to–one mapping $\pi : A \to B$ such that for all $i \in A, P_i = P'_{\pi(i)}$.

Let $\bar{X} = \{\{x, y\} \in X^2 : x \neq y\}$ be the set of all pairs of distinct alternatives. Given the preference profile $P_N \in \mathcal{P}^N$ and a pair of alternatives $\{x, y\} \in \bar{X}$, $\#(xP_Ny) = \#\{i \in N : xP_iy\}$ denotes the number of individuals who prefer x to y at P_N. Then, $n_{x,y}(P_N) = |\#(xP_Ny) - \#(yP_Nx)|$ is the absolute difference between the number of individuals who prefer x to y and the ones who prefer y to x, always at P_N. For all preference profiles $P_N \in \mathcal{P}^N$, let $d_j(P_N) = \#\{\{x, y\} \in \bar{X} : n_{x,y}(P_N) = j\}$ be the number of pairs of alternatives such that $n_{x,y}(P_N)$ is equal to j. It is easy to see that $d_j(P_N)$ may only take strictly positive values for $j = 0, 2, \ldots, n$ whenever the size of the society is even and for $j = 1, 3, \ldots, n$ whenever n is odd. Consequently, the notation $d(P_N) = (d_0(P_N), d_2(P_N), \ldots, d_n(P_N))$ is used for any even n and $d(P_N) = (d_1(P_N), d_3(P_N), \ldots, d_n(P_N))$ for any odd n.

Given the preference relation $P_i \in \mathcal{P}$ and the pair of alternatives $\{x, y\} \in \bar{X}$, we will say that the ordered pair (x, y) is a *contiguous pair at* P_i if xP_iy and there is no other alternative $z \in X$ such that xP_izP_iy. For any two preference relations $P_i, P'_i \in \mathcal{P}$ and any pair of alternatives $\{x, y\} \in \bar{X}$, P'_i is said to be (x, y)–*different from* P_i if (y, x) is a contiguous pair at P_i, (x, y) is a contiguous pair at P'_i, and $zP_iw \Leftrightarrow zP'_iw$ for all pairs of alternatives $\{z, w\} \neq \{x, y\}$. Finally, given the preference profiles $P_N, P'_N \in \mathcal{P}^N$, an individual $i \in N$, and the pair of alternatives $\{x, y\} \in \bar{X}$, P'_N is said to be (x, y)–*different from* P_N *for individual* i if P'_i is (x, y)–different from P_i and $P'_j = P_j$ for all $j \neq i$. Intuitively, P'_N is (x, y)–different from P_N for i if P'_N can be derived from P_N by only reversing the binary relation yP_ix. In this case, we will also say that P'_N can be obtained from P_N by means of a yP_ix*–change*.

Given a set of individuals $N \subset \mathbb{N}$, a *consensus measure for society* N is a function M^N :

$\mathcal{P}^{\mathbb{N}} \to [0,1]$ that assigns to every profile $P \in \mathcal{P}^{\mathbb{N}}$ a real number $M^N(P)$ from the unit interval, with the property that for all preference profiles $P, P' \in \mathcal{P}^{\mathbb{N}}$ such that $P'_N = P_N, M^N(P') = M^N(P)$. For notational purposes, we will write $M(P_N)$ instead of $M^N(P)$. A *consensus measure* M is a family of functions $\{M^N : \mathcal{P}^{\mathbb{N}} \to [0,1]\}_{N \subset \mathbb{N}}$.

3 A Linear and Additive Class of Measures

In continuation, we present four basic properties and show that they fully characterize a general class of linear and additive measures. The first property, Anonymity, states that the preferences of all individuals are equally important in determining the consensus in the society. This idea is modelled by assuming that the consensus measure takes equal values for any two isomorphic preference profiles.

ANONYMITY (ANO): The consensus measure M is *anonymous* if for all societies $A, B \subset \mathbb{N}$ and all isomorphic preference profiles $P_A \in \mathcal{P}^A$ and $P'_B \in \mathcal{P}^B$,

$$M(P'_B) = M(P_A).$$

The second property, Independence of Irrelevant Pairs, is an adoption of Arrow's [3] Independence of Irrelevant Alternatives axiom to our setting. It states that the effect on the consensus of a yP_ix–change is independent of the individual preferences over all other binary comparisons. To express this more formally, consider any two preference profiles P_N and $\bar{P}_N$ with the property that no individual changes her/his preferences on the pair $\{x, y\}$ across the two situations. Suppose also that (y, x) is a contiguous pair of alternatives for i at both preference profiles. If individual i only changes her/his opinion over the comparison between x and y, the consensus should change in both situations by the same amount.

INDEPENDENCE OF IRRELEVANT PAIRS (IIP): The consensus measure M is *independent of irrelevant pairs* if for all societies $N \subset \mathbb{N}$, all individuals $i \in N$, all pairs of distinct alternatives $\{x, y\} \in \bar{X}$, and all preference profiles $P_N, P'_N, \bar{P}_N, \bar{P}'_N \in \mathcal{P}^N$ such that $xP_jy \Leftrightarrow x\bar{P}_jy$ for all $j \in N$, P'_N is (x, y)–different from P_N for individual i, and $\bar{P}'_N$ is (x, y)–different from $\bar{P}_N$ for individual i,

$$M(\bar{P}'_N) - M(\bar{P}_N) = M(P'_N) - M(P_N).$$

According to the third property, Neutrality, the consensus measure is not biased with respect to certain alternatives. Formally, if it is possible to derive one preference profile from a distinct one by relabelling alternatives, then the consensus should be the same in both situations.

NEUTRALITY (NEU): The consensus measure M is *neutral* if for all societies $N \subset \mathbb{N}$ and all preference profiles $P_N, P'_N \in \mathcal{P}^N$ such that there exists a permutation $\mu : X \to X$ with the property that for all individuals $i \in N$ and all pairs of distinct alternatives $\{x, y\} \in \bar{X}$, $xP_iy \Leftrightarrow \mu(x)P'_i\mu(y)$,

$$M(P'_N) = M(P_N).$$

The last property, Monotonicity, regards the situation when a single individual i performs a yP_ix–change. Then, if the number of individuals in the rest of the society who prefer x to y is the same as the number of individuals who prefer y to x, the consensus should not vary. If, however, there were more individuals who prefer x to y, the consensus should increase. Monotonicity therefore highlights that the effect on the consensus of interchanging a contiguous pair of alternatives should depend on the preferences the other members of the society have on that pair of alternatives. If the rest of the society has a split opinion, this change should not affect the degree of consensus. Otherwise, the change in the consensus depends on which of the two alternatives is preferred by the majority.

MONOTONICITY (MON): The consensus measure M is *monotone* if for all societies $N \subset \mathbb{N}$, all individuals $i \in N$, all pairs of distinct alternatives $\{x, y\} \in \bar{X}$, and all preference profiles $P_N, P'_N \in \mathcal{P}^N$ such that P'_N is (x, y)–different from P_N for individual i,

$$n_{x,y}(P_{N\setminus\{i\}}) = 0 \Rightarrow M(P'_N) = M(P_N) \text{ and}$$

$$\#(xP_{N\setminus\{i\}}y) > \#(yP_{N\setminus\{i\}}x) \Rightarrow M(P'_N) > M(P_N).$$

Theorem 1 shows that these properties fully characterize a class of linear and additive consensus measures with the vector of the absolute differences $d(P_N)$ as main component. Note that because of Anonymity, the weighting vector a^n depends on the size of the society N and not on the society itself.

Theorem 1 *The following two statements are equivalent:*

a. The consensus measure M satisfies Anonymity, Independence of Irrelevant Pairs, Neutrality, and Monotonicity.

b. For all societies $N \subset \mathbb{N}$, there exists a vector $a^n = (a_1^n, a_2^n, \ldots, a_{[\frac{n}{2}]+1}^n)$, with $a_i^n \in [0, \frac{2}{k(k-1)}]$ for all $i \in \{1, 2, ..., [\frac{n}{2}] + 1\}$ and $a_j^n < a_{j+1}^n$ for all $j \in \{1, 2, ..., [\frac{n}{2}]\}$, such that for all preference profiles $P_N \in \mathcal{P}^N$,

$$M(P_N) = a^n \cdot d(P_N).$$

Proof: It is straightforward to see that the family of consensus measures defined in (b) satisfies the set of properties in (a). Therefore, we consider from now on the other implication. Take any consensus measure M that satisfies ANO, IIP, NEU, and MON. First, we show that for all societies $N \subset \mathbb{N}$, M^N is a function of $d(P_N)$; that is, we are going to prove that for any two preference profiles $P_N, P'_N \in \mathcal{P}^N$ such that $d(P_N) = d(P'_N)$, $M(P_N) = M(P'_N)$.

Take any society $N \subset \mathbb{N}$ and any two preference profiles $P_N, P'_N \in \mathcal{P}^N$ with the property that $d(P_N) = d(P'_N)$. Now, consider any unanimous preference profile $\bar{P}_N \in \mathcal{P}^N$ (*i.e.*, for all $i, j \in N, \bar{P}_i = \bar{P}_j$) and observe that it is possible to arrive from P_N (and P'_N, respectively) to the preference profile $\bar{P}_N$ by means of successive changes of contiguous pairs of alternatives. Then, given any pair of distinct alternatives $\{x, y\} \in \bar{X}$, suppose without loss of generality that $\#(xP_Ny) \geq \#(yP_Nx)$. In continuation, there are two possibilities: $x\bar{P}_iy$ for all $i \in N$ or $y\bar{P}_ix$ for all $i \in N$.

1. If $x\bar{P}_iy$ for all $i \in N$, in the process of arriving from P_N to $\bar{P}_N$, the contiguous pair of alternatives (y, x) has to be changed exactly $\#(yP_Nx)$–times to (x, y). It follows from MON that each of these changes raises the consensus. By IIP and ANO, each incremental only depends on the number of individuals that prefer x to y in the profile at which this change is applied. Formally, denote by $t_j(x, y)$ the incremental in the consensus when a contiguous pair (y, x) is changed to (x, y) and there were j individuals that prefer x to y. Also by NEU, $t_j(x, y) = t_j(z, w) \equiv t_j$ for all ordered pairs of alternatives $(x, y), (z, w)$. Observe that the the total incremental in the consensus from the changes of the contiguous pair (y, x) to (x, y) is $T_{x,y} = \sum_{j=\#(xP_Ny)}^{n-1} t_j$.

2. On the other hand, if $y\bar{P}_ix$ for all $i \in N$, in the process of arriving from P_N to $\bar{P}_N$, the total incremental in the consensus from the changes of the contiguous pair (x, y) to

(y,x) equals $T_{y,x} = \sum_{j=\#(yP_Nx)}^{n-1} t_j$. By IIP and NEU, $t_j + t_{n-1-j} = 0$ for all $j \neq \frac{n-1}{2}$ and, by MON, $t_{\frac{n-1}{2}} = 0$ when n is odd. Consequently, $T_{y,x} = T_{x,y}$, which states that the total incremental in the consensus for the changes in the distinct pair of alternatives $\{x,y\} \in \bar{X}$ in the process of arriving from P_N to $\bar{P}_N$ is independent on which of the two alternatives is preferred at any $\bar{P}_i$.

Since $d(P_N) = d(P'_N)$ by assumption, there exists a one–to–one mapping $\phi : \bar{X} \to \bar{X}$ such that $n_{\phi(\{x,y\})}(P'_N) = n_{x,y}(P_N)$. We can thus conclude that there exists a pair of alternatives $\phi(\{x,y\})$ such that all changes of this pair in the process of arriving at $\bar{P}_N$ from P'_N raises the consensus in the same quantity as changes of the pair of alternatives $\{x,y\}$ in the process of arriving at $\bar{P}_N$ from P_N. Repeating this process for all elements of $\bar{X}$, one can conclude that $M(\bar{P}_N) - M(P'_N) = M(\bar{P}_N) - M(P_N)$. Hence, $M(P'_N) = M(P_N)$.

It remains to be shown that M^N is a linear and additive function of $d(P_N)$. To do that, observe that MON and IIP imply some restrictions on M^N depending on the values of $d(P_N)$. In particular, MON implies that for all preference profiles $P_N, P'_N \in \mathcal{P}^N$ such that for some $i \leq n-2$, $d_i(P'_N) = d_i(P_N) - 1$, $d_{i+2}(P'_N) = d_{i+2}(P_N) + 1$ and $d_j(P'_N) = d_j(P_N)$ for all $j \neq \{i, i+2\}$, $M(P'_N) > M(P_N)$. This implication is called *Condition 1* from now on. Furthermore, IIP implies that for all preference profiles $P_N, P'_N, \bar{P}_N, \bar{P}'_N \in \mathcal{P}^N$ such that for some $i \leq n-2$, $d_i(P'_N) = d_i(P_N) - 1, d_{i+2}(P'_N) = d_{i+2}(P_N) + 1, d_i(\bar{P}'_N) = d_i(\bar{P}_N) - 1$, $d_{i+2}(\bar{P}'_N) = d_{i+2}(\bar{P}_N) + 1$, $d_j(P'_N) = d_j(P_N)$ and $d_j(\bar{P}'_N) = d_j(\bar{P}_N)$ for all $j \neq \{i, i+2\}$, $M(P'_N) - M(P_N) = M(\bar{P}'_N) - M(\bar{P}_N)$. We refer to this implication as *Condition 2*. In continuation, we proceed by induction separating the proof depending on whether the size of the society is even or odd.

1. Suppose that n is even. It follows directly from Condition 1 that M^N would attain its minimum at any preference profile $P_N \in \mathcal{P}^N$ such that $d_0(P_N) = \frac{k(k-1)}{2}$ and $d_j(P_N) = 0$ for all $j \neq 0$. It is easy to see that this type of preference profile exists. An example is a preference profile such that $\frac{n}{2}$ individuals have any preference relation $P_i \in \mathcal{P}$ and $\frac{n}{2}$ individuals have the preference relation $P_j \in \mathcal{P}$ such that for all pairs of alternatives $\{x,y\} \in \bar{X}$, $xP_iy \Leftrightarrow yP_jx$. Without loss of generality, let the consensus at these profiles be $a_1^N \cdot \frac{k(k-1)}{2}$. Consider now any profile P'_N such that $d_i(P'_N) = 0$ for all $i > 2$. Then, by Condition 2, $M(P'_N) = a_1^N \cdot \frac{k(k-1)}{2} + d_2(P'_N) \cdot t_{\frac{n}{2}}$. Fix $a_2^N = a_1^N + t_{\frac{n}{2}}$.

Suppose now that for any preference profile P''_N such that $d_i(P''_N) = 0$ for all $i > q$, with q being an even number in $[4, n-2]$, $M(P''_N) = a_1^N \cdot \frac{k(k-1)}{2} + d_2(P''_N) \cdot t_{\frac{n}{2}} + \ldots + d_q(P''_N) \cdot t_{\frac{n+q-2}{2}}$ and that $a^N_{\frac{q}{2}+1} = a^N_{\frac{q}{2}} + t_{\frac{n+q-2}{2}}$. Next, consider any preference profile $\tilde{P}_N$ such that $d_i(\tilde{P}_N) = 0$ for all $i > q+2$. Then, by Condition 2, we have that $M(\tilde{P}_N) = a_1^N \cdot \frac{k(k-1)}{2} + d_2(\tilde{P}_N) \cdot t_{\frac{n}{2}} + \ldots + d_q(\tilde{P}_N) \cdot t_{\frac{n+q-2}{2}} + d_{q+2}(\tilde{P}_N) \cdot t_{\frac{n+q}{2}}$. Define $a^N_{\frac{q}{2}+2} = a^N_{\frac{q}{2}+1} + t_{\frac{n+q}{2}}$. Hence, the vector a^N is completely defined and it is easy to see that for any arbitrary preference profile $P_N \in \mathcal{P}^N$, $M(P_N) = a^N \cdot d(P_N)$.

2. Suppose that n is odd. It follows directly from Condition 1 that M^N would attain its minimum at any preference profile $P_N \in \mathcal{P}^N$ such that $d_1(P_N) = \frac{k(k-1)}{2}$ and $d_i(P_N) = 0$ for all $i \neq 1$. It is easy to see that this type of preference profile exists. An example is a preference profile such that $\frac{n-1}{2}$ individuals have any preference relation $P_i \in \mathcal{P}$ and $\frac{n+1}{2}$ individuals have the preference relation $P_j \in \mathcal{P}$ such that for all pairs of distinct alternatives $\{x, y\} \in \bar{X}$, $xP_iy \Leftrightarrow yP_jx$. Without loss of generality, let the consensus at these profiles be $a_1^N \cdot \frac{k(k-1)}{2}$. Consider now any profile P'_N such that $d_i(P'_N) = 0$ for all $i > 3$. Then, by Condition 2, $M(P'_N) = a_1^N \cdot \frac{k(k-1)}{2} + d_3(P'_N) \cdot t_{\frac{n+1}{2}}$. Fix $a_2^N = a_1^N + t_{\frac{n+1}{2}}$.
   Suppose now that for any preference profile P''_N such that $d_i(P''_N) = 0$ for all $i > q$, with q being an odd number in $[5, n-2]$, $M(P''_N) = a_1^N \cdot \frac{k(k-1)}{2} + d_3(P''_N) \cdot t_{\frac{n+1}{2}} + \ldots + d_q(P''_N) \cdot t_{\frac{n+q-2}{2}}$ and that $a^N_{\frac{q+1}{2}} = a^N_{\frac{q-1}{2}} + t_{\frac{n+q-2}{2}}$. Next, consider any preference profile $\tilde{P}_N$ such that $d_i(\tilde{P}_N) = 0$ for all $i > q+2$. Then, by Condition 2, we have that $M(\tilde{P}_N) = a_1^N \cdot \frac{k(k-1)}{2} + d_3(\tilde{P}_N) \cdot t_{\frac{n+1}{2}} + \ldots + d_q(\tilde{P}_N) \cdot t_{\frac{n+q-2}{2}} + d_{q+2}(\tilde{P}_N) \cdot t_{\frac{n+q}{2}}$. Let $a^N_{\frac{q+3}{2}} = a^N_{\frac{q+1}{2}} + t_{\frac{n+q}{2}}$. Hence, the vector a^N is completely defined and it is easy to see that for any arbitrary preference profile $P_N \in \mathcal{P}^N$, $M(P_N) = a^N \cdot d(P_N)$.

It follows from ANO that $a^N = a^{\bar{N}}$ whenever the societies N and $\bar{N}$ are equally sized. Therefore, it can be concluded that for any arbitrary preference profile $P_N \in \mathcal{P}^N$, $M(P_N) = a^n \cdot d(P_N)$. By Condition 1, $t_i > 0$ for all $i \geq \frac{n}{2}$. This implies that for all n and all $j \in \{1, 2, ..., [\frac{n}{2}]\}, a_j^n < a_{j+1}^n$. Given also that the minimal value of M^N is attained at all preference profiles $P_N \in \mathcal{P}^N$ such that $d_0(P_N) = \frac{k(k-1)}{2}$ if n is even or $d_1(P_N) = \frac{k(k-1)}{2}$ if n is odd, and by definition this minimal value has to be non–negative, we get that $a_1^n \geq 0$ for all n. Finally, by Condition 1, the maximal value of M^N would be attained at all preference profiles $P_N \in \mathcal{P}^N$ such that $d_n(P_N) = \frac{k(k-1)}{2}$. Since these preference profiles exist (*i.e.*, the unanimous preference profiles P_N such that $xP_iy \Leftrightarrow xP_jy$ for all $i, j \in N$ and all $\{x, y\} \in \bar{X}$) and, by definition, the maximal value of M^N cannot be greater than 1, the restriction $a^n_{[\frac{n}{2}]+1} \leq \frac{2}{k(k-1)}$ has to be satisfied. This concludes the proof of the theorem. □

We denote the class of consensus measures characterized in Theorem 1 by Ω. We conclude this section with a graphical representation showing that all measures belonging to Ω establish a *partial order* on preference profiles of the same size that is of the same nature as first order stochastic dominance and Lorenz curve domination (if $n \leq 3$, this order is complete). The following example will illustrate this point.

Suppose that $n = 8$ and $k = 5$ so that, in total, there are ten different pairs of alternatives. In Figure 1, the x–axis refers to the indexes of the vector d, which are $0, 2, 4, 6$ and 8. The y–axis, on the other hand, collects cumulative percentages of d represented by the function $D(P_N, j)$. This function assigns to any preference profile $P_N \in \mathcal{P}^N$ and any number $j \in \{0, 2, 4, 6, 8\}$, the value $D(P_N, j)$ such that $\frac{2}{k(k-1)} \cdot \#\{\{x, y\} \in \bar{X} : n_{x,y}(P_N) \leq j\} = D(P_N, j)$. For example, if $D(P_N, 2) = 0.6$, as it is the case of profile 3, $n_{x,y}(P_N)$ is smaller than or equal to 2 for sixty percent of all possible pairs of alternatives.

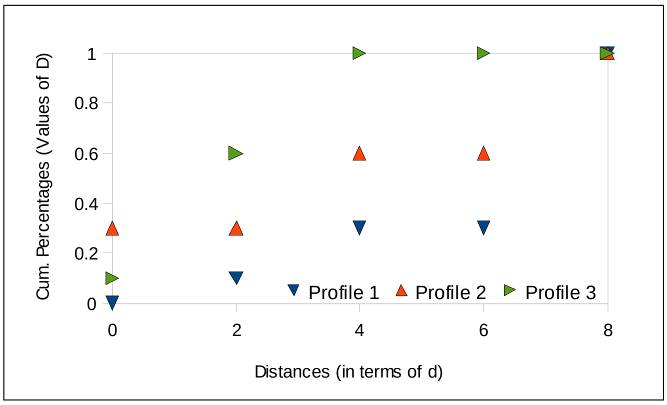

Figure 1: Dominance criterion.

Now, if the values of D of one preference profile lie never above and in some case strictly below the ones of a second preference profile, all measures belonging to Ω assign a higher consensus to the former profile. Consequently, given the three preference profiles in Figure 1, the consensus in profile 1 is higher than in profiles 2 and 3 for all measures in Ω. Note also that the measures belonging to Ω do not put any restriction on which preference profile has the higher consensus whenever the values of D cross (as it is the case for profiles 2 and

3). In fact, it can be easily shown that in these situations, there are always three consensus measures in Ω such that one measure assigns a higher consensus to the first preference profile, another measure gives a higher consensus to the second preference profile, and a third measure imposes that both preference profiles have the same consensus.

There are many ways to complete the partial order induced by Ω. One intuitive approach is to calculate, for any two preference profiles, the distances between the values of D. The distances should then be summed up in order to decide which of the two preference profiles has the higher consensus. This idea will motivate a particular consensus measure, which we will call *average sigma*, whose exact formulation will be introduced in Section 5.

4 The Family Γ

Theorem 1 shows that a set of intuitive axioms restricts the way in which the consensus should be measured; it exactly imposes a linear and additive formula over the majority results in the pairwise comparisons over alternatives. In addition, it also induces some ordinal comparisons between preference profiles of the same size. However, the class Ω is rather large and includes measures that suffer from several shortcomings. In particular:

1. So far, the structure introduced is not sufficient to establish undoubtable comparisons between preference profiles of different sizes, even in ordinal terms. For example, Ω contains measures that assign less consensus to an unanimous preference profile of a society with size n than to a preference profile of individuals with very different opinions in a society of size $n+1$. Some structure on the values of a consensus measure for preference profiles of different sizes is needed.

2. A particular characteristic of the family Ω is that the exact cardinal values the measure should take are not determined. For example, some consensus measures of Ω only take values between 0 and ε (for any $\varepsilon > 0$), whereas others only take values between $1-\varepsilon$ and 1. Ω even includes measures that take values between 0 and ε for some group sizes and values between $1-\varepsilon$ and 1 for others. This shows why some homogeneous structure is needed so that the value assigned to a preference profile has some meaning *per se*.

3. According to Theorem 1, a yP_ix–change increases the consensus if x wins in a majority voting against y taking into account the preferences of the rest of individuals. It also

allows the quantity of this increase to depend on the size of the majority. However, Ω includes measures that evaluate these effects arbitrarily, for example depending on whether or not the majority in a given preference profile is a prime number. Hence, some consistency in the quantity changes is asked for.

The criticism raised above motivates us to propose additional axioms in order to further restrict Ω. The first axiom, Replication, puts some structure on preference profiles belonging to different societies. It is an invariance property asking that if a society and their preferences are cloned a number of times, the new society, which consists of the union of all cloned groups, should have the same consensus as the original one. It is inspired by the scale invariance axiom of Allison [1] used in characterizations of the Gini index.

REPLICATION (REP): The consensus measure M satisfies *replication* if for all preference profiles $P_A \in \mathcal{P}^A$ and $P'_B \in \mathcal{P}^B$, where $|B| = m \cdot |A|$ (with $m \in \mathbb{N}$), such that P'_B consists of the union of m isomorphic copies of P_A,

$$M(P'_B) = M(P_A).$$

Observe that Replication implies Anonymity for the particular case of $m = 1$. To fill the second gap and construct homogeneous consensus measures, we require that the extreme values a consensus measure can reach by definition (0 and 1) are attained for some preference profiles (corresponding to possibly different societies of possibly different sizes). This property serves the objective that the value a consensus measure assigns to a preference profile provides some information *per se*.

FULL RANGE (FR): The consensus measure M satisfies *full range* if there are two societies A and B and two preference profiles $P_A \in \mathcal{P}^A$ and $P'_B \in \mathcal{P}^B$ such that

$$M(P_A) = 0 \text{ and } M(P'_B) = 1.$$

To introduce the final condition, Consistency, consider a preference profile P_N such that three individuals $i, j, l \in N$ have a contiguous pair (y, x) in their preferences, but there is a majority of individuals that prefer x to y. Now, suppose that individual i changes her/his preference over the pair $\{x, y\}$. We know from Monotonicity that this yP_ix–change increases the consensus by some quantity. Suppose now that, after this yP_ix–change, j also changes her opinion on $\{x,y\}$. As a result, the consensus increases again by some quantity. Finally, if l also switches this preference, the consensus increases again. However, it seems natural that these quantity changes should be related to each other in some way. For example, if the second change increases the consensus in the same quantity as the first one, then there are not apparent reasons why the third quantity change should be different to the other two. If, however, the second change increases the consensus by a larger (smaller) quantity than the first one, then it seems natural that the third change increases the consensus by this amount more (less) than the second one.

CONSISTENCY (CON): The consensus measure M is *consistent* if for all societies $N \subset \mathbb{N}$, all individuals $i,j,l \in N$, all pairs of distinct alternatives $\{x,y\} \in \bar{X}$, and all preference profiles $P_N, P'_N, P''_N, P'''_N \in \mathcal{P}^N$ such that P'_N is (x,y)–different from P_N for i, P''_N is (x,y)–different from P'_N for j, P'''_N is (x,y)–different from P''_N for l, and $\#(xP_Ny) \geq \#(yP_Nx)$,

$$(M(P'_N) - M(P_N)) - (M(P''_N) - M(P'_N)) = (M(P''_N) - M(P'_N)) - (M(P'''_N) - M(P''_N)).$$

Adding Full Range and Consistency to the former properties and replacing Anonymity by Replication characterizes a subfamily of Ω with the particular feature that the election of a consensus measure in this subfamily depends on the value of a single parameter. To introduce this subfamily formally, consider the coefficients $p(n,\gamma)$ and $q(n,\gamma)$ defined for all $n \geq 2$ and all $\gamma \in \mathbb{R}$:

$$p(n,\gamma) = \begin{cases} 16\gamma + (8\gamma - 2)(n-4) & \text{if } n \text{ is even} \\ 4\gamma + (4\gamma - 1)(n-2) & \text{if } n \text{ is odd} \end{cases}$$

and

$$q(n,\gamma) = \begin{cases} 16 - 32\gamma & \text{if } n \text{ is even} \\ 4 - 8\gamma & \text{if } n \text{ is odd.} \end{cases}$$

Now, we are ready to state our main result.

Theorem 2 *The consensus measure M satisfies Independence of Irrelevant Pairs, Neutrality, Monotonicity, Replication, Full Range, and Consistency if, and only if, $M \in \Omega$ and there exists $\gamma \in [\frac{1}{4}, \frac{3}{4}]$ such that for all $n \geq 2$ and all $i \in \{1, ..., [\frac{n}{2}] + 1\}$,*

$$a_i^n = \begin{cases} \frac{i-1}{n^2k(k-1)}\left(2p(n,\gamma) + (i-2)q(n,\gamma)\right) & \textit{if } n \textit{ is even} \\ \frac{2i-1}{n^2k(k-1)}\left(2p(n,\gamma) + (2i-2)q(n,\gamma)\right) & \textit{if } n \textit{ is odd.} \end{cases}$$

Proof: Observe that the family of consensus measures defined in the theorem satisfies the properties. Therefore, we consider from now on the other implication. Take any consensus measure M that satisfies IIP, NEU, MON, REP, FR, and CON. By Theorem 1, $M \in \Omega$. In the following, we investigate the additional restrictions REP, FR, and CON impose on the set of vectors $a \equiv \{a^n\}_{n\geq 2}$. First, we establish six preliminary claims that will help us to develop the proof later on.

Claim 1 *If n is odd, $a_i^n = a_{2i}^{2n}$ for all $i \in \{1, ..., [\frac{n}{2}] + 1\}$.*

Proof: Consider any odd n and any $i \leq [\frac{n}{2}] + 1$. Take any preference profile $P_N \in \mathcal{P}^N$ such that $d_{2i-1}(P_N) = \frac{k(k-1)}{2}$ and $d_j(P_N) = 0$ for all $j \neq 2i-1$. Such a preference profile always exists. For example, it is a preference profile in which $\frac{n-1}{2} + i$ individuals have a preference relation P_i and $\frac{n+1}{2} - i$ have a preference relation P_j such that for all $\{x, y\} \in \bar{X}$, $xP_iy \Leftrightarrow yP_jx$. Then, by Theorem 1, we have that $M(P_N) = a_i^n \cdot \frac{k(k-1)}{2}$. Consider now a society N' of size $2n$ and a preference profile $P'_{N'}$ such that it is the union of 2 isomorphic and disjoint copies of P_N. Then, by definition, $d_{4i-2}(P'_{N'}) = \frac{k(k-1)}{2}$ and $d_j(P'_{N'}) = 0$ for all $j \neq 4i-2$. Then, by Theorem 1, we have that $M(P'_{N'}) = a_{2i}^{2n} \cdot \frac{k(k-1)}{2}$. By REP, $M(P_N) = M(P'_{N'})$. Therefore, $a_i^n = a_{2i}^{2n}$. □

Claim 2 *If n is even, $a_i^n = a_{2i-1}^{2n}$ for all $i \in \{1, ..., [\frac{n}{2}] + 1\}$.*

Proof: Consider any even n and any $i \leq \frac{n}{2}+1$. Take any preference profile $P_N \in \mathcal{P}^N$ such that $d_{2i-2}(P_N) = \frac{k(k-1)}{2}$ and $d_j(P_N) = 0$ for all $j \neq 2i-2$. Such a preference profile always exists. For example, it is a preference profile in which $\frac{n}{2} + i - 1$ individuals have a preference relation P_i and $\frac{n}{2} - i + 1$ have a preference relation P_j such that for all $\{x, y\} \in \bar{X}$, $xP_iy \Leftrightarrow yP_jx$. Then, by Theorem 1, we have that $M(P_N) = a_i^n \cdot \frac{k(k-1)}{2}$. Consider now a society N' of size $2n$ and a preference profile $P'_{N'}$ such that it is the union of 2 isomorphic and disjoint copies of P_N. Then, by definition, $d_{4i-4}(P'_{N'}) = \frac{k(k-1)}{2}$ and $d_j(P'_{N'}) = 0$ for all $j \neq 4i-4$. Then, by Theorem 1, we have that $M(P_N) = a_{2i-1}^{2n} \cdot \frac{k(k-1)}{2}$. By REP, $M(P_N) = M(P'_{N'})$. Therefore, $a_i^n = a_{2i-1}^{2n}$. □

Claim 3 *If n is odd, $a_1^n > 0$.*

<u>Proof</u>: Suppose otherwise. Then, there is some odd n such that $a_1^n = 0$. Then, by Claim 1, $a_2^{2n} = 0$. Since $a_i^t < a_{i+1}^t$ for all t and all $i = 1, \ldots, [\frac{t}{2}]$ by Theorem 1, it must be the case that $a_1^{2n} < 0$. But this is not possible given that $a_1^t \geq 0$ for all t by Theorem 1. $\square$

Claim 4 *If n is even, $a_1^n = 0$.*

<u>Proof</u>: Consider any even n. By FR, there is some society C of size c and some preference profile $P'_C \in \mathcal{P}^C$ such that $M(P'_C) = 0$. It follows from Theorem 1 that the minimal value of M^C is $a_1^c \cdot \frac{k(k-1)}{2}$. Then, it follows from Claim 3 that c is an even number. By definition, P'_C should be one preference profile with the minimal value of M^C. By Theorem 1, this minimal value is attained in preference profiles P''_C satisfying that $d_0(P''_C) = \frac{k(k-1)}{2}$, which always exist. An example is P''_C such that $\frac{c}{2}$ individuals have the preference P''_i and the other $\frac{c}{2}$ individuals have the preference P''_j such that $xP''_i y \Leftrightarrow yP''_j x$ for all $\{x, y\} \in \bar{X}$. By Theorem 1, $M(P''_C) = M(P'_C) = 0$.

Take any pair of individuals $\{i, j\} \subseteq C$ such that $P''_i \neq P''_j$. By REP, $M(P''_{\{i,j\}}) = M(P''_C) = 0$. Since $d_0(P''_{\{i,j\}}) = \frac{k(k-1)}{2}$ and $d_2(P''_{\{i,j\}}) = 0$, we know that $M(P''_{\{i,j\}}) = a_1^2 \cdot \frac{k(k-1)}{2}$ by Theorem 1. Hence, $a_1^2 = 0$. Now, consider the preference profile P_N that consists of the union of $\frac{n}{2}$ isomorphic and disjoint copies of $P''_{\{i,j\}}$. Then, $d_0(P_N) = \frac{k(k-1)}{2}$ and $d_j(P_N) = 0$ for all $j \neq 0$. By Theorem 1, we have that $M(P_N) = a_1^n \cdot \frac{k(k-1)}{2}$. By REP, we have that $M(P_N) = M(P''_{\{i,j\}}) = 0$. Therefore, $a_1^n = 0$. $\square$

Claim 5 *For all n, $a_{[\frac{n}{2}]+1}^n = \frac{2}{k(k-1)}$.*

<u>Proof</u>: Consider any n. Obverse first that, by FR, there exists a society C of size c and a preference profile $P'_C \in \mathcal{P}^C$ for which the consensus is 1. By definition, P'_C should be one preference profile with the maximal value of M^C. By Theorem 1, this maximal value is attained in preference profiles P''_C satisfying that $d_c(P''_C) = \frac{k(k-1)}{2}$ and $d_i(P''_C) = 0$ for all $i \neq c$, which always exist. An example is P''_C such that all individuals $i \in C$ have the same preference relation P''_i. By Theorem 1, $M(P'_C) = M(P''_C) = a_{[\frac{c}{2}]+1}^c \cdot \frac{k(k-1)}{2} = 1$. Therefore, we have that $a_{[\frac{c}{2}]+1}^c = \frac{2}{k(k-1)}$. Consider now the profile $\bar{P}_B$ consisting in the union of n isomorphic and disjoint copies of P'_C. Then, $d_{nc}(\bar{P}_B) = \frac{k(k-1)}{2}$ and $d_i(\bar{P}_B) = 0$ for all $i \neq nc$. By REP, $M(\bar{P}_B) = M(P'_C) = 1$. Therefore, $a_{[\frac{nc}{2}]+1}^{nc} = \frac{2}{k(k-1)}$. Consider the profile P_N of size n such

that all individuals $i \in N$ have the preference relation P_i. Note that $d_n(P_N) = \frac{k(k-1)}{2}$ and $d_i(P_N) = 0$ for all $i \neq n$. Then, $M(P_N) = a^n_{[\frac{n}{2}]+1} \cdot \frac{k(k-1)}{2}$ and $\bar{P}_B$ consists in the union of c isomorphic and disjoint copies of P_N. Then, by REP, $M(P_N) = M(\bar{P}_B) = 1$. Therefore, $a^n_{[\frac{n}{2}]+1} = \frac{2}{k(k-1)}$. □

Claim 6 *If n is a multiple of 4, $a^4_2 = a^n_{\frac{n}{4}+1}$.*

Proof: Consider any n multiple of 4, a society C of size 4 and a preference profile $P'_C \in \mathcal{P}^C$ such that $d_2(P'_C) = \frac{k(k-1)}{2}$ and $d_j(P'_C) = 0$ for all $j \neq 2$. Then, $M(P'_C) = a^4_2 \cdot \frac{k(k-1)}{2}$. Consider now the preference profile P_N of size n consisting in the union of $\frac{n}{4}$ isomorphic and disjoint copies of P'_C. Note that $d_{\frac{n}{2}}(P_N) = \frac{k(k-1)}{2}$ and $d_j(P_N) = 0$ for all $j \neq \frac{n}{2}$. Then, $M(P_N) = a^n_{\frac{n}{4}+1} \cdot \frac{k(k-1)}{2}$. By REP, $M(P_N) = M(P'_C)$. Therefore, $a^4_2 = a^n_{\frac{n}{4}+1}$. □

To find the exact description of the set of vectors a, we divide the proof in two parts, depending on whether the size of the society N is even or odd.

1. Suppose that n is even. We know from Claims 4 and 5 that $a^n_1 = 0$ and $a^n_{\frac{n}{2}+1} = \frac{2}{k(k-1)}$. This concludes the proof for the case of $n = 2$. Therefore, suppose in continuation that $n \geq 4$. If $n = 4$, the unique value not determined by the axioms is a^4_2. Define $\gamma \equiv a^4_2 \cdot \frac{k(k-1)}{2}$ and observe that, for the moment, by Theorem 1, $\gamma \in (0, 1)$. If $n \geq 6$, it follows from CON that for all for all $i \in \{1, \ldots, \frac{n}{2} - 2\}$, $\big((a^n_{i+2} - a^n_{i+1}) - (a^n_{i+1} - a^n_i)\big) = \big((a^n_{i+3} - a^n_{i+2}) - (a^n_{i+2} - a^n_{i+1})\big)$. Consequently, if we define $\bar{p}(n, \gamma) \equiv a^n_2$ and $\bar{q}(n, \gamma) \equiv \big((a^n_{i+2} - a^n_{i+1}) - (a^n_{i+1} - a^n_i)\big)$ for any $i \in \{1, \ldots, \frac{n}{2} - 2\}$, we have that $a^n_i = (i-1)\bar{p}(n, \gamma) + \frac{(i-2)(i-1)}{2}\bar{q}(n, \gamma)$ for all $i \in \{1, \ldots, \frac{n}{2} + 1\}$.

   In case n is a multiple of 4, $a^n_{\frac{n}{4}+1} = a^4_2 = \gamma \cdot \frac{2}{k(k-1)}$ by Claim 6. Consequently, we obtain the following set of linear equations:

$$\begin{aligned} \tfrac{n}{2} \cdot \bar{p}(n, \gamma) + \tfrac{\frac{n}{2}(\frac{n}{2}-1)}{2} \cdot \bar{q}(n, \gamma) &= \tfrac{2}{k(k-1)} \\ \tfrac{n}{4} \cdot \bar{p}(n, \gamma) + \tfrac{\frac{n}{4}(\frac{n}{4}-1)}{2} \bar{q}(n, \gamma) &= \gamma \cdot \tfrac{2}{k(k-1)}. \end{aligned}$$

   Solving for $\bar{p}(n, \gamma)$ and $\bar{q}(n, \gamma)$ yields

$$\begin{aligned} \bar{p}(n, \gamma) &= \tfrac{2}{n^2 k(k-1)}(16\gamma + (8\gamma - 2)(n-4)) \\ \bar{q}(n, \gamma) &= \tfrac{2}{n^2 k(k-1)}(16 - 32\gamma). \end{aligned}$$

Letting $p(n,\gamma) = \frac{n^2k(k-1)}{2}\bar{p}(n,\gamma)$ and $q(n,\gamma) = \frac{n^2k(k-1)}{2}\bar{q}(n,\gamma)$, one can easily check that if n is a multiple of 4, then for all $i \in \{1,\dots,\frac{n}{2}+1\}$,

$$a_i^n = \frac{i-1}{n^2k(k-1)}\left(2p(n,\gamma)+(i-2)q(n,\gamma)\right).$$

For all other even values of n, observe that $a_i^n = a_{2i-1}^{2n}$ by Claim 2. Then, given that $2n$ is a multiple of 4, we have that for all $i \in \{1,\dots,\frac{n}{2}+1\}$, $a_i^n = \frac{2i-2}{(2n)^2k(k-1)}(2\cdot(16\gamma+(8\gamma-2)(2n-4))+(2i-3)(16-32\gamma))$. It is then easy to deduce the result.

2. Suppose now that n is odd. We know from Claim 1 that $a_i^n = a_{2i}^{2n}$. Consequently, we have that for all $i \in \{1,\dots,[\frac{n}{2}]+1\}$, $a_i^n = \frac{2i-1}{(2n)^2k(k-1)}(2(16\gamma+(8\gamma-2)(2n-4))+(2i-2)(16-32\gamma))$. It is then easy to deduce the result.

Finally, it remains to be shown that $\gamma \in [\frac{1}{4},\frac{3}{4}]$. We know from Theorem 1 that for all n and all $i \in \{2,\dots,[\frac{n}{2}]+1\}$, it is necessary that $a_i^n - a_{i-1}^n > 0$. To guarantee it, we can focus on the cases in which n is even and strictly greater than 2.[1] We have then that for all $i \in \{2,\dots,\frac{n}{2}+1\}$, $a_i^n - a_{i-1}^n = \bar{p}(n,\gamma)+(i-2)\cdot\bar{q}(n,\gamma)$. Since this equation reduces to $a_2^n - a_1^n = \bar{p}(n,\gamma)$ for $i=2$, it is necessary that $p(n,\gamma) > 0$ for all n even and strictly greater than 2. Consequently, it is required that $16\gamma+(8\gamma-2)(n-4) > 0$ for all n even and strictly greater than 2. It is easy to see that if this inequality holds for $n \to \infty$, it also holds for all n even and strictly greater than 2. For $n \to \infty$ this equation reduces to $8\gamma - 2 \geq 0$, which is equivalent to $\gamma \geq \frac{1}{4}$.

To establish that $a_i^n - a_{i-1}^n > 0$ for all $i \in \{3,\dots,\frac{n}{2}+1\}$, we only have to show that $a_{\frac{n}{2}+1}^n - a_{\frac{n}{2}}^n = \bar{p}(n,\gamma)+(\frac{n}{2}-1)\bar{q}(n,\gamma) > 0$. This is because $\bar{p}(n,\gamma) > 0$ for $\gamma \geq \frac{1}{4}$ and for any negative $\bar{q}(n,\gamma)$, $a_i^n - a_{i-1}^n$ is minimized for $i = \frac{n}{2}+1$. Using some algebra, it can be verified that the necessary condition is $n(6-8\gamma)+8(2\gamma-1) > 0$. Again, it is sufficient to check the condition for $n \to \infty$. For $n \to \infty$ this equation reduces to $6 - 8\gamma \geq 0$, which is equivalent to $\gamma \leq \frac{3}{4}$. This concludes the proof of the Theorem. □

We denote the class of consensus measures characterized in Theorem 2 by Γ. With respect to the parameter γ, it follows from the proof that its exact formulation is given by $\gamma =$

[1] To see why suppose that n is odd and that $a_i^n < a_{i-1}^n$. Then, by Claim 2, $a_{2i}^{2n} < a_{2i-2}^{2n}$. Hence, the condition is also violated for any even n. For the special case when $n=2$, we already know that $a_1^2 = 0$ and $a_2^2 = 1$.

$a_2^4 \cdot \frac{k(k-1)}{2}$. Hence its value is related to the question of how much consensus there is in a society of four individuals if, for all pairs of alternatives, three individuals prefer one alternative and one individual the other. Observe also that by Theorem 1, any change of a contiguous pair of alternatives produces a variation in the consensus whose sign depends on the alternative that wins in a majority election when the electorate is formed by all individuals but the one who changes her/his opinion. However, the magnitude of this variation is allowed to depend on the size of the majority and it is this dependence that is now uniquely captured by the parameter γ. If $\gamma = 0.5$, all changes in favor of the winning alternative are considered equally important; that is, the consensus measure associated with this value ignores the size of the majority. If $\gamma > 0.5$, the effect on the consensus of a change in a contiguous pair is the higher the more divided the society is on that particular pair of alternatives (and this effect is more pronounced the higher γ is). Finally, if $\gamma < 0.5$, a change of a contiguous pair gets a higher weight the less divided the society is in their opinions on these two alternatives (and this effect is more pronounced the smaller γ is).

5 The Focal Measures of Γ

In this section, we analyze some particular measures of Γ; in particular, we concentrate on the measures that are identified by $\gamma = 0.25$ and $\gamma = 0.5$. These measures have not only a special interest because they are focal, but also because they have an intuitive interpretation. We will also see that the third focal case, $\gamma = 0.75$, is intimately related to the one of $\gamma = 0.25$.

5.1 Corrected Normalized Average Tau

For the special case of two individuals ($N = \{i, j\}$), Kemeny [13] and Kendall [12] have proposed, in different contexts, a correlation (distance) measure that has received particular attention. To define it, take any preference profile $P_N \in \mathcal{P}^N$ as given. Now, for any pair of alternatives $\{x, y\} \in \bar{X}$, $\tau_{i,j}^{x,y}(P_N) = 1$ if individual i and j agree on the binary ordering between x and y, otherwise $\tau_{i,j}^{x,y}(P_N) = 0$. The consensus in this society, $\tau_{i,j}(P_N)$, is then defined as the percentage of pairwise comparisons that both individuals agree upon; that is,

$\tau_{i,j}(P_N) = \frac{2}{k(k-1)} \sum_{\{x,y\}\in \bar{X}} \tau_{i,j}^{x,y}(P_N)$.[2] Hays [11] suggests to naturally extend Kendall's τ to the case of any arbitrary number of individuals by calculating the average of all $\tau_{i,j}(P_N)$ for all pairs of individuals in the society; that is, for all societies $N \subset \mathbb{N}$ and all preference profiles $P_N \in \mathcal{P}^N$, $\bar{\tau}(P_N) = \frac{2}{n(n-1)} \sum_{i,j\in N: j\neq i} \tau_{i,j}(P_N)$. This measure is referred to as *average tau.* Since the extreme values of $\bar{\tau}^N$ for a given society N may be different from 0 and 1, it can worthwhile to normalize the measure to the unit interval. The *normalized average tau* is given by $\tilde{\tau}(P_N) = \frac{\bar{\tau}(P_N) - min\{\bar{\tau}^N\}}{max\{\bar{\tau}^N\} - min\{\bar{\tau}^N\}}$, taking into account that $max\{\bar{\tau}^N\} = 1$, $min\{\bar{\tau}^N\} = \frac{n-2}{2(n-1)}$ if n is even, and $min\{\bar{\tau}^N\} = \frac{n-1}{2n}$ in case n is odd.

We have argued before that all measures belonging to Γ only differ in how the consensus varies as a response to a yP_ix–change. This magnitude is, in each measure, a function of $n_{x,y}$. By definition, average tau and its normalized version evaluate a change of a consecutive pair depending on the variations in the distances between the preference ranking of this individual and the preference rankings of each of the other individuals. To be more exact, a yP_ix–change produces two effects on $\bar{\tau}$ and $\tilde{\tau}$: first, it reduces the distance (and, thus, increases the consensus) between the preference of this individual and the ones of the individuals who prefer the alternative that is made better off and, second, it increases the distance (and, thus, reduces the consensus) between the preference of this individual and the ones of the individuals who prefer the other alternative. As a result, the increase in the consensus changes proportionally with the number of individuals that prefer the alternative that is made better off. This is formally expressed by the following property.

PROPORTIONALITY (PROP): The consensus measure M is *proportional* if for all societies $N \subset \mathbb{N}$, all individuals $i \in N$, all pairs of distinct alternatives $\{x, y\} \in \bar{X}$, and all preference profiles $P_N, P'_N, \bar{P}_N, \bar{P}'_N \in \mathcal{P}^N$ such that P'_N is (x, y)–different from P_N for individual i, $\bar{P}'_N$ is (x, y)–different from $\bar{P}_N$ for individual i, $\#(xP_Ny) \geq \#(yP_Nx)$, and $\#(x\bar{P}_Ny) \geq \#(y\bar{P}_Nx)$,

$$\frac{M(P'_N) - M(P_N)}{M(\bar{P}'_N) - M(\bar{P}_N)} = \frac{n_{x,y}(P_{N\setminus\{i\}})}{n_{x,y}(\bar{P}_{N\setminus\{i\}})}.$$

One would think that adding Proportionality to the former properties leads to a charac-

[2] This is the exact formulation of Kendall [12]. The formulation of Kemeny [13] is in terms of absolute distance, but its adaption to relative terms leads to the very same formulation.

terization of average tau or its normalized version. However, this is not true. The reason is that these measures do not satisfy Replication. To see why, consider the preference profile $P_{\{1,2,3\}}$ such that individual 1 and 2 have identical preference rankings that are completely opposed to the one of individual 3. Consider now the preference profile $P_{\{1,2,3,4,5,6\}}$ such that $P_1 = P_2 = P_4 = P_5$ and $P_3 = P_6$. Replication implies that the consensus is in both situations the same. However, $\bar{\tau}(P_{\{1,2,3\}}) = \frac{1}{3}$ and $\bar{\tau}(P_{\{1,2,3,4,5,6\}}) = \frac{7}{15}$, whereas $\tilde{\tau}(P_{\{1,2,3\}}) = 0$ and $\tilde{\tau}(P_{\{1,2,3,4,5,6\}}) = \frac{1}{9}$.

Consequently, the following question arises: if the idea of $\bar{\tau}$ and $\tilde{\tau}$ (represented by PROP) is natural and the axioms that characterize Γ are also desirable (in particular, REP), which is the measure of Γ that satisfies PROP? We are going to see that it corresponds with the value $\gamma = 0.25$. Curiously, it coincides with $\tilde{\tau}$ for even societies but corrects it for odd societies in such a way that the deviation tends to zero as the size of the society goes to infinity. In particular, for any preference profile P_N belonging to a society N with an odd number of individuals, the measure can be calculated as $\frac{1}{n^2} + \tilde{\tau}(P_N)\frac{n^2-1}{n^2}$.

Definition 1 The consensus measure M is called *corrected normalized average tau ($\hat{\tau}$)* if $M \in \Gamma$ and $\gamma = 0.25$ or, equivalently, if for all $n \geq 2$ and all $j \in \{1, \dots, [\frac{n}{2}] + 1\}$,

$$a_j^n = \begin{cases} \frac{8(j-1)^2}{n^2 k(k-1)} & \text{if } n \text{ is even} \\ \frac{2(2j-1)^2}{n^2 k(k-1)} & \text{if } n \text{ is odd} \end{cases}.$$

Replacing IIP and CON by PROP isolates $\bar{\tau}$ from all measures in Γ.

Theorem 3 *The consensus measure M satisfies Replication, Neutrality, Proportionality, Monotonicity, and Full Range if, and only if, it is $\hat{\tau}$.*

Proof: It is straightforward to see that $\hat{\tau}$ satisfies the set of axioms. To show the other implication, consider any consensus measure M that satisfies REP, NEU, PROP, MON, and FR. Since IIP and CON are implied by the other five properties, Theorem 2 applies and we only have to determine $\gamma = a_2^4 \cdot \frac{k(k-1)}{2}$. Since it follows from Theorem 2 that $a_1^4 = 0$ and $a_3^4 = \frac{2}{k(k-1)}$, we can apply PROP to obtain that $\frac{\frac{2}{k(k-1)}\gamma}{1-\frac{2}{k(k-1)}\gamma} = \frac{1}{3}$. This equation solves for $\gamma = 0.25$. This concludes the proof of the theorem. $\square$

Observe that the measure $\gamma = 0.75$ is obtained by inverting PROP. However, since this measure does not have an intuitive explanation, we abstain from presenting its characterization formally.

5.2 Average Sigma

While the average tau calculates the distances between individual preferences, an alternative approach determines the distances in pairwise comparisons of alternatives according to the majority rule. For the special case when there are only two alternatives –*i.e.*, $X = \{x, y\}$– the approach consists in calculating the absolute difference in the support of the two alternatives and weighting it by the maximal difference possible; that is, for all $N \subset \mathbb{N}$ and all preference profiles $P_N \in \mathcal{P}^N$, $\sigma_{x,y}(P_N) = \frac{n_{x,y}(P_N)}{n}$. This measure can be naturally extended to the case of an arbitrary number of alternatives by following the reasoning of Hays [11] and calculating the average of all $\sigma_{x,y}(P_N)$ for all pairs of alternatives. Formally, this implies that $\bar{\sigma}(P_N) = \frac{2}{k \cdot (k-1)} \sum_{\{x,y\} \in \bar{X}} \sigma_{x,y}(P_N) = \frac{2}{k \cdot n \cdot (k-1)} \sum_{\{x,y\} \in \bar{X}} n_{x,y}(P_N)$. We will call this measure *average sigma*. In terms of the set of vectors a, the following equivalent definition is obtained.

Definition 2 The consensus measure M is called *average sigma ($\bar{\sigma}$)* if $M \in \Gamma$ and $\gamma = 0.5$ or, equivalently, if for all $n \geq 2$ and all $j \in \{1, \ldots, [\frac{n}{2}] + 1\}$,

$$a_j^n = \begin{cases} \frac{4(j-1)}{nk(k-1)} & \textit{if } n \textit{ is even} \\ \frac{4j-2}{nk(k-1)} & \textit{if } n \textit{ is odd} \end{cases}.$$

An interpretation of average sigma in terms of Γ is straightforward: if the majority of the society prefer x to y, any yP_ix–change induces the consensus measure to vary in the same way, independently of the size of the majority. With respect to Figure 1, it implies that a complete ordering on preference profiles of societies with the same size is obtained by summing up the distances in the values of the function D. Formally, this can be expressed as follows: the effect of a interchange of a contiguous pair of alternatives only depends on which alternative had the majority before the change.

STRONG INDEPENDENCE OF IRRELEVANT PAIRS (SIIP): The consensus measure M is *strongly independent of irrelevant pairs* if for all societies $N \subset \mathbb{N}$, all individuals $i \in N$, all pairs of distinct alternatives $\{x, y\} \in \bar{X}$, and all preference profiles $P_N, P'_N, \bar{P}_N, \bar{P}'_N \in$

$\mathcal{P}^N$ such that P'_N is (x, y)–different from P_N for individual i, $\bar{P}'_N$ is (x, y)–different from $\bar{P}_N$ for individual i, $\#(xP_Ny) \geq \#(yP_Nx)$, and $\#(x\bar{P}_Ny) \geq \#(y\bar{P}_Nx)$,

$$M(\bar{P}'_N) - M(\bar{P}_N) = M(P'_N) - M(P_N).$$

Replacing IIP and CON by SIIP isolates $\bar{\sigma}$ from all measures belonging to Γ.

Theorem 4 *The consensus measure M satisfies Replication, Strong Independence of Irrelevant Pairs, Neutrality, Monotonicity, and Full Range if, and only if, it is $\bar{\sigma}$.*

Proof: It is straightforward to see that $\bar{\sigma}$ satisfies the set of axioms. To show the other implication, consider any consensus measure M that satisfies REP, SIIP, NEU, MON, and FR. Since IIP and CON are implied by the other five properties, Theorem 2 applies and we only have to determine $\gamma = a_2^4 \cdot \frac{k(k-1)}{2}$. Since it follows from Theorem 2 that $a_1^4 = 0$ and $a_3^4 = \frac{2}{k(k-1)}$, we can apply SIIP to obtain that $a_2^4 - 0 = \frac{2}{k(k-1)} - a_2^4$. Hence, $a_2^4 = \frac{1}{2} \cdot \frac{2}{k(k-1)}$, which is equivalent to $\gamma = 0.5$. This concludes the proof of the Theorem. □

6 Final Remarks

The objective of this paper was to analyze consensus measures from a normative point of view. We first introduced a set of basic properties (Anonymity, Independence of Irrelevant Pairs, Neutrality, and Monotonicity) and showed that they fully characterize a family of linear and additive measures Ω that takes the vector of distances d as main component. In the next step of our study, we used the properties of Replication, Full Range, and Consistency to restrict Ω further. In particular, the three additional properties imply that the set of weighting vectors a depends on a single parameter (γ), which determines the extend to which the size of the majority in a pairwise comparison affects the changes in the consensus. Finally, we showed that the polar cases of Γ have natural explanations. In this sense, we regard our analysis as a first contribution to understand the nature of consensus measures. In continuation, we will draw the attention to different approaches to operationalize this concept.

The Nature of Consensus Measures

The question of how to construct a social welfare function –that is, how to aggregate individual preferences into a social preference relation– is at the heart of social choice theory. This question can to large extend be reduced to the discussion of how the social ordering between any two alternatives (say x and y) should be determined. The first proposal goes back to Condorcet [8] who advocated that the social preference between x and y should be determined by focusing only on the opinions individuals have on the pair $\{x, y\}$ thereby ignoring any other considerations. Arrow [3] formalized this idea in his famous Independence of Irrelevant Alternatives axiom. This view was above all challenged by Borda [5] who proposed that the positions the alternatives occupy in the individual rankings should be taken into account.

In our problem of constructing a consensus measure, the very same debate arises: should the effect of a yP_ix–change on the consensus depend on how individuals rank other pairs of alternatives? Defenders of Condorcet's position would give a negative answer to this question and require that the axiom of Independence of Irrelevant Alternatives is adapted to the current context. We follow this approach by imposing Independence of Irrelevant Pairs. An alternative way that definitively remains to be explored is to study consensus measures that explicitly consider the position of the alternatives in the rankings. Some measures applied in statistics (without axiomatic analysis) can be situated in this group, for example Kendall's coefficient of concordance.

Apart from this classical discussion, following the view of Condorcet has one remarkable advantage: while Arrow [3] showed that it is impossible to construct a sensible social welfare function on the universal preference domain that is Pareto efficient and satisfies Independence of Irrelevant Alternatives, we are here able to come up with positive answers assuming Independence of Irrelevant Pairs.

Social Choice versus Social Orders

We have implicitly assumed throughout that the objective of the group is to rank alternatives. This approach has to be carefully distinguished from the situation when the society wants to select one or more alternatives, as it typically occurs for example in presidential elections. In that case, if all members of the society agree on which alternatives have to be chosen, differences at the bottom of the preferences do not matter. One possible way to generalize the measures presented here into that direction is to modify IIP to allow that higher ranked yP_ix–changes receive more weight. An example is the family M_3 presented in the Appendix where the independence of the properties is studied. In continuation, we concretize this idea.

Example 1. Assume that $X = \{x, y, z\}$ and $N = \{1, 2, 3, 4\}$ and let the preference profiles P_N and P'_N be as follows.

P_1	P_2	P_3	P_4
x	x	x	x
y	y	y	y
z	z	z	z

P'_1	P'_2	P'_3	P'_4
x	x	x	x
y	y	z	z
z	z	y	y

All measures belonging to Γ assign a consensus of 1 to P_N and a consensus strictly less than 1 to P'_N. However if the objective of the society is to select a single alternative, all members agree to select x in both situation and, therefore, the consensus should be 1 under both preference profiles. One way to incorporate that idea is to restrict the attention to the information provided by the top alternative, which is equivalent to assign a weight of zero to all changes of preferences that take place between the second and third alternative of a given preference relation. □

Consensus in Terms of Polarization

We have postulated implicitly that the consensus measures are inversely related to the probability of conflicts in groups. While this relation remains to be tested empirically, we would like to mention that the concept of *polarization* has recently gained prominence over the one of inequality in the literature on conflicts and group cohesion (see, Esteban and Ray [9] and Duclos et al. [10]). The concluding example will show the relation between our measures and the concept of polarization.

Example 2. Assume that $X = \{x, y, z\}$ and $N = \{1, 2, 3, 4\}$ and let the preference profiles P_N and P'_N be as follows.

P_1	P_2	P_3	P_4
x	x	z	z
y	y	y	y
z	z	x	x

P'_1	P'_2	P'_3	P'_4
x	y	z	z
y	x	x	y
z	z	y	x

All measures belonging to Γ assign zero consensus to the two preference profiles. However, it can be argued that polarization is higher in the first situation, because under P_N, there are two natural subgroups of equal sizes with opposed opinions (individuals 1 and 2 versus individuals 3 and 4). This is usually the situation where polarization is maximal and this fact is captured by our measures. Such a division cannot be found under P'_N but our measures assign nevertheless the minimal value also to these preference profiles. Some refinements of our measures should be necessary to differentiate between these preference profiles. □

References

[1] Allison, P. (1978). Measures of Inequality. *American Sociological Review* 43, 865-880.

[2] Atkinson, A. (1970). On the Measurement of Inequality. *Journal of Economic Theory* 2, 244-263.

[3] Arrow, K. (1963). Social Choice and Individual Values. 2nd edition, John Wiley, New York.

[4] Bosch, R. (2006). Characterizations of Voting Rules and Consensus Measures. Ph. D. Dissertation, Tilburg University.

[5] Borda, J. (1781). Mémoire sur les Élections au Scrutin. *Histoire de l'Academie Royale des Sciences*, Paris.

[6] Cook, W. and Seiford, L. (1978). Priority Ranking and Consensus Formation. *Management Science* 24, 1721–1732.

[7] Cook, W. and Seiford, L. (1982). On the Borda–Kendall Consensus Method for Priority Ranking Problems. *Management Science* 28, 621–637.

[8] Condorcet, M. (1785). An Essay on the Application of Probability Decision Making: An Election between three Candidates. In: The Political Theory of Condorcet (eds. Sommerlad, F. and McLean, I.), University of Oxford, Oxford, 1989.

[9] Esteban, J. and Ray, D. (1994). On the Measurement of Polarization. *Econometrica* 62, 819–852.

[10] Duclos, J., Esteban, J., and Ray, D. (2006). Polarization: Concepts, Measurement, Estimation. *Econometrica* 74, 1737–1772.

[11] Hays, W. (1960). A Note on Average Tau as a Measure of Concordance. *Journal of the American Statistical Association* 55, 331–341.

[12] Kendall, M. (1962). Rank Correlation Methods, 3rd edition, Hafner Publishing Company, New York.

[13] Kemeny, J. (1959). Mathematics Without Numbers. *Daedalus* 88, 577–591.

[14] Klamler, C. (2008). A Distance Measure for Choice Functions. *Social Choice and Welfare* 30, 419–425, 2008.

[15] Meskanen, T. and Nurmi, H. (2007). Distance from Consensus: A Theme and Variations. In: Mathematics and Democracy. Recent Advances in Voting Systems and Collective Choice (eds.: Simeone, B., Pukelsheim, F.), pp. 117–132, Springer, Heideberg.

[16] Rothschild, M. and Stiglitz, J. (1969). Increasing Risk: A Definition and its Economic Consequences, Cowles Foundation Discussion Paper 275.

Appendix

Independence in Theorem 1

We show by means of four examples that the properties in Theorem 1 are independent.

Anonymity: For every society $N \subset \mathbb{N}$, assign to each group of individuals $A \subseteq N$ a non–negative number $p_N(A)$ such that $p_N(A) = 0$ if $\#A \leq \frac{n}{2}$, $p_N(A) > p_N(B)$ if $\#A > \#B$, $p_N(N) = \frac{2}{k(k-1)}$, and $p_{N'}(A) \neq p_{N'}(B)$ for some society $N' \subset \mathbb{N}$ and some $A, B \subset N'$ such that $A \neq B$ and $\#A = \#B$. Given a preference profile P_N and a pair of alternatives $\{x, y\} \in \bar{X}$, denote the value assigned to the maximal set of individuals who prefer the alternative that is favored by the majority as $s_{x,y}(P_N) = p_N(\{A \subseteq N : \#(xP_Ay) \geq \#(yP_Ax)$ and there is no $B \subseteq N$ s.t. $\#(xP_By) > \#(xP_Ay)\})$. Now, let consensus measure M_1 be such that for all societies N and all preference profiles P_N, $M_1(P_N) = \sum_{\{x,y\} \in \bar{X}} s_{x,y}(P_N)$. This consensus measure satisfies NEU, IIP, and MON. The following example shows that it is not anonymous. Let $N = \{1, 2, 3\}$ and $X = \{x, y\}$. Suppose that the preference profiles P_N and P'_N are such that xP_1y, xP_2y, yP_3x, yP'_1x, xP'_2y, and xP'_3y. Moreover, let

$p_N(\{1,2\}) = p_N(\{1,3\}) = \frac{1}{2}$ and $p_N(\{2,3\}) = \frac{3}{4}$. Then, $M_1(P_N) = \frac{1}{2}$ and $M_1(P'_N) = \frac{3}{4}$. ANO would imply that $M_1(P_N) = M_1(P'_N)$.

Neutrality: Let $q : \bar{X} \to \mathbb{R}_{++}$ be a function that assigns to each pair of alternatives $\{x, y\} \in \bar{X}$ a strictly positive weight $q_{x,y} > 0$ in such a way that $q_{x,y} \neq q_{w,z}$ for some $\{w, z\} \neq \{x, y\}$. Now, let consensus measure M_2 be such that for all societies N and all preference profiles P_N, $M_2(P_N) = \sum_{\{x,y\}\in\bar{X}} \frac{q_{x,y}}{\sum_{\{w,z\}\in\bar{X}} q_{w,z}} \sigma_{x,y}(P_N)$. This consensus measure satisfies ANO, IIP, and MON. The following example shows that it is not neutral. Let $N = \{1, 2\}$ and $X = \{x, y, z\}$. Suppose that the preference profiles P_N and P'_N are such that xP_1yP_1z, yP_2xP_2z, $zP'_1yP'_1x$, and $yP'_2zP'_2x$. Moreover, let $q_{x,z} = q_{y,z} = 1$ and $q_{x,y} = 2$. Then, $M_2(P_N) = \frac{2}{4}$ and $M_2(P'_N) = \frac{3}{4}$. NEU would imply that $M_2(P_N) = M_2(P'_N)$.

Independence of Irrelevant Pairs: Let $f : \{1, \ldots, k\} \to \mathbb{R}_{++}$ be a function that assigns to each position in a ranking a strictly positive weight such that $f(v) \neq f(w)$ for some $v, w \in \{1, \ldots, k\}$. Then, for each society $N \subset \mathbb{N}$, define the function $F_N : X \times \mathcal{P}^N \to \mathbb{R}_{++}$ as $F_N(x, P_N) = \sum_{i\in N} f(\#\{y \in X : \neg yP_ix\})$. Now, let the consensus measure M_3 be such that for all societies N and all preference profiles P_N, $M_3(P_N) = \sum_{\{x,y\}\in\bar{X}} \frac{F_N(x,P_N)+F_N(y,P_N)}{\sum_{\{w,z\}\in\bar{X}} F_N(w,P_N)+F_N(z,P_N)} \cdot \sigma_{x,y}(P_N)$. This consensus measure satisfies ANO, NEU, and MON. The following example shows that it is not independent of irrelevant pairs. Let $N = \{1, 2, 3\}$ and $X = \{x, y, z\}$. Suppose that the preference profiles P_N and $\bar{P}_N$ are such that xP_1yP_1z, $P_2 = P_1$, yP_3xP_3z, $z\bar{P}_1x\bar{P}_1y$, $\bar{P}_2 = \bar{P}_1$, and $z\bar{P}_3y\bar{P}_3x$. Let the preference profiles P'_N and $\bar{P}'_N$ be obtained by performing a yP_3x–change and a $y\bar{P}_3x$–change, respectively. Moreover, let $f(1) = 3$, $f(2) = 2$ and $f(3) = 1$. Then $M_3(P'_N) - M_3(P_N) = 1 - \frac{2}{3} = \frac{1}{3}$ and $M_3(\bar{P}'_N) - M_3(\bar{P}_N) = 1 - \frac{8}{9} = \frac{1}{9}$. IIP would imply that $M_3(P'_N) - M_3(P_N) = M_3(\bar{P}'_N) - M_3(\bar{P}_N)$.

Monotonicity: Let the consensus measure M_4 be such that for all societies N and all preference profiles P_N, $M_4(P_N) = 1 - \bar{\sigma}(P_N)$. This consensus measure satisfies ANO, NEU, and IIP. The following example shows that it is not monotone. Let $N = \{1, 2\}$ and $X = \{x, y\}$. Suppose that the preference profiles P_N and P'_N are such that xP_1y, yP_2x, xP'_1y, and xP'_2y. Then, $M_4(P_N) = 1$ and $M_4(P'_N) = 0$. MON would imply that $M_4(P'_N) > M_4(P_N)$.

Independence in Theorem 2

We show by means of six examples that the properties in Theorem 2 are independent.

Replication: Let the consensus measure M_5 be such that for all societies N and all preference profiles P_N, $M_5(P_N) = \bar{\tau}(P_N)$. This consensus measure satisfies NEU, IIP, MON, FR, and CON. The following example shows that it does not satisfy Replication. Let $N = \{1, 2, 3\}$ and $\bar{N} = \{1, 2, 3, 4, 5, 6\}$. Take any P_N such that individual 1 and 2 have identical preference rankings that are completely opposed to the one of individual 3. Then, $M_5(P_N) = \frac{1}{3}$. If $P_4 = P_5 = P_1$ and $P_6 = P_3$, $M_5(P_{\bar{N}}) = \frac{7}{15}$. Replication would imply that $M_5(P_{\bar{N}}) = M_5(P_N)$.

Neutrality: The consensus measure M_2 satisfies REP, IIP, MON, FR, and CON. However, it is not neutral.

Independence of Irrelevant Pairs: The consensus measure M_3 satisfies REP, NEU, MON, FR, and CON. However, it is not independent of irrelevant pairs.

Monotonicity: The consensus measure M_4 satisfies REP, NEU, IIP, FR, and CON. However, it is not monotone.

Full Range: Let the consensus measure M_6 be such that for all societies N and all preference profiles P_N, $M_6(P_N) = \frac{1}{2} \cdot \bar{\sigma}(P_N)$. This consensus measure satisfies REP, NEU, IIP, MON, and CON. Since its maximum is $\frac{1}{2}$, it does not satisfy Full Range.

Consistency: Let the consensus measure $M_7 \in \Omega$ be such that for all n and all $j \in \{1, 2, ..., [\frac{n}{2}] + 1\}$,

$$a_j^n = \begin{cases} \frac{2(2^{j-1}-1)}{k(k-1)(2^{n/2}-1)} \text{ if } n \text{ is even} \\ \frac{2(2^{2j-1}-1)}{k(k-1)(2^n-1)} \text{ if } n \text{ is odd} \end{cases}.$$

This consensus measure satisfies REP, NEU, IIP, MON, and FR. The following example shows that it is not consistent. Suppose that $N = \{1, 2, 3, 4, 5, 6\}$ and $X = \{x, y\}$. Let P_N be such that xP_1y, yP_4x, $P_1 = P_2 = P_3$, and $P_4 = P_5 = P_6$. Then, $M_7(P_N) = 0$. If the preference profile P'_N is (x, y)–different from P_N for 4, then $M_7(P'_N) = \frac{1}{7}$. If the preference profile P''_N is (x, y)–different from P'_N for individual 5, then $M_7(P''_N) = \frac{3}{7}$. Finally, if the preference profile P'''_N is (x,y)–different from P''_N for individual 6, then $M_7(P'''_N) = 1$. Consequently, $(M_7(P''_N) - M_7(P'_N)) - (M_7(P'_N) - M_7(P_N)) = \frac{2}{7} - \frac{1}{7} = \frac{1}{7}$, whereas $(M_7(P'''_N) - M_7(P''_N)) - (M_7(P''_N) - M_7(P'_N)) = \frac{4}{7} - \frac{2}{7} = \frac{2}{7}$. Consistency would imply that $(M_7(P''_N) - M_7(P'_N)) - (M_7(P'_N) - M_7(P_N)) = (M_7(P'''_N) - M_7(P''_N)) - (M_7(P''_N) - M_7(P'_N))$.

Independence in Theorem 3

We show by means of five examples that the properties in Theorem 3 are independent.

Replication: The consensus measure M_5 satisfies NEU, PROP, MON, and FR. However, it does not satisfy Replication.

Neutrality: Let $q : \bar{X} \to \mathbb{R}_{++}$ be a function that assigns to each pair of alternatives $\{x,y\} \in \bar{X}$ a strictly positive weight $q_{x,y} > 0$ in such a way that $q_{x,y} \neq q_{w,z}$ for some $\{w,z\} \neq \{x,y\}$. Given a society N, a preference profile P_N, and two distinct individuals $i,j \in N$, let $w_{i,j}(P_N) = \sum_{\{x,y\}\in\bar{X}} \frac{q_{x,y}}{\sum_{\{w,z\}\in\bar{X}} q_{w,z}} \cdot \tau^{x,y}_{i,j}(P_N)$ be the weighted percentage of pairwise comparisons individual i and j agree upon. Define $\bar{w}(P_N) = \frac{2}{n(n-1)} \sum_{i,j\in N: j\neq i} w_{i,j}(P_N)$ as the average weighted tau. Now, let consensus measure M_8 be such that for all societies N and all preference profiles P_N, $M_8(P_N) = \frac{\bar{w}(P_N) - \frac{1}{n}}{\frac{n-1}{n}}$ if n is even and $M_8(P_N) = \frac{1}{n^2} + \frac{\bar{w}(P_N) - \frac{n-1}{2n}}{\frac{n+1}{2n}} \cdot \frac{n^2-1}{n^2}$ if n is odd. This consensus measure satisfies REP, PROP, MON, and FR. The following example shows that it is not neutral. Let $N = \{1,2\}$ and $X = \{x,y,z\}$. Suppose that the preference profiles P_N and P'_N are such that xP_1yP_1z, yP_2xP_2z, $zP'_1yP'_1x$, and $yP'_2zP'_2x$. Moreover, let $q_{x,z} = q_{y,z} = 1$ and $q_{x,y} = 2$. Then, $M_8(P_N) = \frac{2}{4}$ and $M_8(P'_N) = \frac{3}{4}$. NEU would imply that $M_8(P_N) = M_8(P'_N)$.

Proportionality: Let the consensus measure M_9 be such that for all societies N and all preference profiles P_N, $M_9(P_N) = \bar{\sigma}(P_N)$. This consensus measure satisfies REP, NEU, MON, and FR. The following example shows that it is not proportional. Let $N = \{1,2,3,4\}$ and $X = \{x,y\}$. Suppose that the preference profiles P_N and $\bar{P}_N$ are such that xP_1y, yP_2x, $P_3 = P_1$, $P_4 = P_2$, $\bar{P}_1 = \bar{P}_2 = \bar{P}_3 = P_1$, and $\bar{P}_4 = P_2$. Let the preference profiles P'_N and $\bar{P}'_N$ be obtained by performing a yP_4x–change and a $y\bar{P}_4x$–change, respectively. Then, $M_9(P'_N) - M_9(P_N) = \frac{1}{2} - 0 = \frac{1}{2}$ and $M_9(\bar{P}'_N) - M_9(\bar{P}_N) = 1 - \frac{1}{2} = \frac{1}{2}$. PROP would imply that $\frac{M_9(P'_N) - M_9(P_N)}{M_9(\bar{P}'_N) - M_9(\bar{P}_N)} = \frac{1}{3}$.

Monotonicity: Let the consensus measure M_{10} be such that for all societies N and all preference profiles P_N, $M_{10}(P_N) = 1 - \hat{\tau}(P_N)$. This consensus measure satisfies REP, NEU, PROP, and FR. The following example shows that it is not monotone. Let $N = \{1, 2\}$ and $X = \{x, y\}$. Suppose that the preference profiles P_N and P'_N are such that xP_1y, yP_2x, xP'_1y, and xP'_2y. Then, $M_{10}(P_N) = 1$ and $M_{10}(P'_N) = 0$. MON would imply that $M_{10}(P'_N) > M_{10}(P_N)$.

Full Range: Let the consensus measure M_{11} be such that for all societies N and all preference profiles P_N, $M_{11}(P_N) = \frac{1}{2} \cdot \hat{\tau}(P_N)$. This consensus measure satisfies REP, NEU, PROP, and MON. Since its maximum is $\frac{1}{2}$, it does not satisfy Full Range.

Independence in Theorem 4

We show by means of five examples that the properties in Theorem 4 are independent.

Replication: Let the consensus measure M_{12} be such that for all societies N and all preference profiles P_N, $M_{12}(P_N) = \bar{\sigma}(P_N)$ whenever n is even and $M_{12}(P_N) = \frac{\bar{\sigma}(P_N) - \frac{1}{n}}{1 - \frac{1}{n}}$ if n is odd. This consensus measure satisfies NEU, SIIP, MON, and FR. The following example shows that it does not satisfy Replication. Let $N = \{1, 2, 3\}$, $\bar{N} = \{1, 2, 3, 4, 5, 6\}$, and $X = \{x, y\}$. Let xP_1y, $yP_2x, P_3 = P_4 = P_6 = P_1$, and $P_5 = P_2$. Then, $M_{12}(P_N) = 0$ and $M_{12}(P_{\bar{N}}) = \frac{1}{3}$. Replication would imply that $M_{12}(P_{\bar{N}}) = M_{12}(P_N)$.

Neutrality: The consensus measure M_2 satisfies REP, SIIP, MON, and FR. However, it is not neutral.

Strong Independence of Irrelevant Pairs: Let the consensus measure M_{13} be such that for all societies N and all preference profiles P_N, $M_{13}(P_N) = \hat{\tau}(P_N)$. This consensus measure satisfies REP, NEU, MON, and FR. The following example shows that it is not strongly independent of irrelevant pairs. Let $N = \{1, 2, 3, 4\}$ and $X = \{x, y\}$. Suppose that the preference profiles P_N and $\bar{P}_N$ are such that xP_1y, yP_2x, $P_3 = P_1$, $P_4 = P_2$, $\bar{P}_1 = \bar{P}_2 = \bar{P}_3 = P_1$, and $\bar{P}_4 = P_2$. Let the preference profiles P'_N and $\bar{P}'_N$ be obtained by performing a yP_4x–change and a $y\bar{P}_4x$–change, respectively. Then, $M_{13}(P'_N) - M_{13}(P_N) = \frac{1}{4} - 0 = \frac{1}{4}$ and $M_{13}(\bar{P}'_N) - M_{13}(\bar{P}_N) = 1 - \frac{1}{4} = \frac{3}{4}$. SIIP would imply that $M_{13}(P'_N) - M_{13}(P_N) = M_{13}(\bar{P}'_N) - M_{13}(\bar{P}_N)$.

Monotonicity: The consensus measure M_4 satisfies REP, NEU, SIIP, and FR. However, it does not satisfy Monotonicity.

Full Range: The consensus measure M_6 satisfies REP, NEU, SIIP, and MON. However, it does not satisfy Full Range.